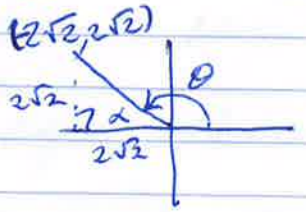
Finding Square Roots of Complex Numbers with De Moivre's Theorem

① Find the square roots of $2\sqrt{2}(-1+i)$

Solution : Method 1

$2\sqrt{2}(-1+i) = -2\sqrt{2} + i2\sqrt{2}$



$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$

$\theta = \frac{3\pi}{4}$

Modulus of $2\sqrt{2}(-1+i)$

$|2\sqrt{2}(-1+i)| = 4$

$\therefore 2\sqrt{2}(-1+i) = 4\,\text{cis}\left(\frac{3\pi}{4} + 2k\pi\right)$

Find all z: $\boxed{z^2 = 4\,\text{cis}\left(\frac{3\pi}{4} + 2k\pi\right)}$ equate

$z = r\,\text{cis}\,\theta \rightarrow \boxed{z^2 = r^2\,\text{cis}\,2\theta}$

$\therefore r^2 = 4 \rightarrow r = 2 \rightarrow |z| = 2$

$\therefore 2\theta = \frac{3\pi}{4} + 2k\pi \rightarrow \theta = \frac{3\pi}{8} + k\pi$

$\therefore z = 2\,\text{cis}\left(\frac{3\pi}{8} + k\pi\right)$

There are 2 distinct solutions
$\therefore$ use $k = 0, 1$

$\therefore z_0 = 2\,\text{cis}\,\frac{3\pi}{8}, \quad z_1 = \frac{11\pi}{8}$ ✓

Solution: Method 2

Same as Method 1 up to

$z = 2\,\text{cis}\,\frac{3\pi}{4}$ without the $2k\pi$

Now, recall when taking the square root of a number we use $\pm$

e.g. $\sqrt{4} = \pm 2$

$\therefore$ for $z = 2\sqrt{2}(-1+i) \Rightarrow z = 4\,\text{cis}\left(\frac{3\pi}{4}\right)$

$\therefore \boxed{\sqrt{z} = \pm\left(4\,\text{cis}\,\frac{3\pi}{4}\right)^{\frac{1}{2}}} = \pm 2\,\text{cis}\,\frac{3\pi}{8}$

$\Rightarrow 2\,\text{cis}\,\frac{3\pi}{8}, \; -2\,\text{cis}\,\frac{3\pi}{8}$

Note: This is NOT $2\,\text{cis}\left(-\frac{3\pi}{8}\right)$

$-2\,\text{cis}\,\frac{3\pi}{8}$

$= -2\cos\frac{3\pi}{8} - i\,2\sin\frac{3\pi}{8}$

Why do the two methods produce different answers?

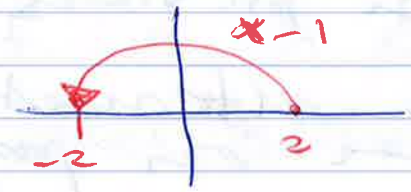
Method 1: $2\,\text{cis}\,\frac{3\pi}{8}$, $2\,\text{cis}\,\frac{11\pi}{8}$

Method 2: $2\,\text{cis}\,\frac{3\pi}{8}$, $-2\,\text{cis}\,\frac{3\pi}{8}$

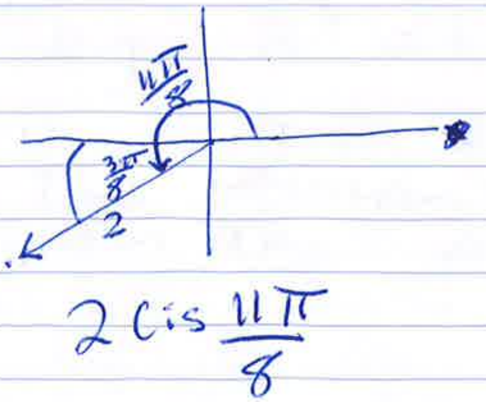
How can $2\,\text{cis}\,\frac{11\pi}{8}$ and $-2\,\text{cis}\,\frac{3\pi}{8}$ be equivalent.

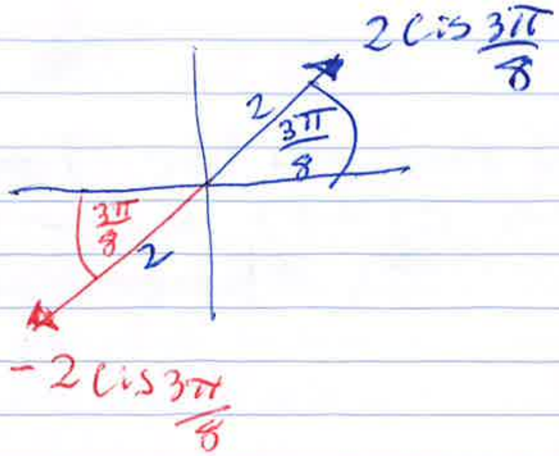
Recall that multiplying by i is a rotation of $\frac{\pi}{2}$ and multiplying by -1 is a rotation of π.

e.g.



$\therefore$ $-1 \times 2\,\text{cis}\,\frac{3\pi}{8}$ is a rotation of $\text{cis}\,\frac{3\pi}{8}$ by a factor of π





Geometrically you can see they are equivalent.