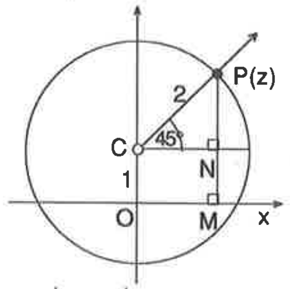
Solution 4.9(6): $|z - i| = 2$ and $\arg(z - i) = \frac{\pi}{4}$



The locus of $|z - i| = 2$ is a circle of radius 2 with the centre at C(0, 1).
Arg $(z - i)$ represents a ray through C, making an angle of 45°. These two loci intersect in only one point P(x, y).

$CP = 2, \angle PCN = 45°$

$x = OM = CN = 2\cos 45° = \sqrt{2}$

$y = PM = PN + NM = 2\sin 45° + 1 = \sqrt{2} + 1$

$\therefore$ P is $(\sqrt{2}, \sqrt{2} + 1)$.

SOLUTIONS OF PROBLEMS: CHAPTER 5

SOLUTIONS: EXERCISE 5.1

Solution 5.1(1): (a) $16x^4 - 1$
$= (4x^2 - 1)(4x^2 + 1)$
$= (2x - 1)(2x + 1)(2x + i)(2x - i)$

(b) $4x^2 + x + 3 = 4(x^2 + \frac{x}{4} + \frac{3}{4})$

$= 4[(x + \frac{1}{8})^2 + \frac{47}{64}]$

$= 4(x + \frac{1}{8} + i\frac{\sqrt{47}}{8})(x + \frac{1}{8} - i\frac{\sqrt{47}}{8})$

(c) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$
$= (x^2 + 1)^2 - x^2$
$= (x^2 + x + 1)(x^2 - x + 1)$

$= [(x + \frac{1}{2})^2 + \frac{3}{4}][(x - \frac{1}{2})^2 + \frac{3}{4}]$

$= (x + \frac{1}{2} + \frac{\sqrt{3}}{2}i)(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i)\times$

$\times(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i)(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i)$

Solution 5.1(2): We use the Remainder theorem: When P(x) is divided by x – a, the remainder is P(a).

$P(x) = 2x^4 + x^2 - i = x^2(2x^2 + 1) - i$

When P(x) is divided by $x - (1 - 2i)$ the remainder is $P(1 - 2i)$.

Now $x = 1 - 2i$

$\therefore$ $x^2 = 1 - 4i + 4i^2 = -3 - 4i$

$\therefore$ $P(1 - 2i) = (-3 - 4i)(-6 - 8i + 1) - i$
$= (3 + 4i)(5 + 8i) - i$
$= 15 + 44i + 32i^2 - i = -17 + 43i$

Solution 5.1(3): Let $P(x) = x^4 + bx^3 + cx^2 - x + 2$
Since P(x) is divisible by $x^2 - 1$, it is divisible by each of the factors of $x^2 - 1$, i.e. $(x - 1)$ and $(x + 1)$. We use the Factor theorem: If $(x - a)$ is a factor of P(x) then P(a) = 0. Substituting x = 1

$P(1) = 1 + b + c - 1 + 2 = 0$

$\therefore$ $b + c = -2$ (1)

Again substituting $x = -1$,

$P(-1) = 1 - b + c + 1 + 2 = 0$

$\therefore$ $-b + c = -4$ (2)

Adding (1) and (2): $2c = -6$, giving $c = -3$
Then from (1): $b - 3 = -2$, giving $b = 1$
Hence b = 1 and c = –3.

Solution 5.1(4): When P(x) is divided by $(x - 3)(x - 4)$, the remainder must be of the form ax + b.

$\therefore$ $P(x) = (x - 3)(x - 4)Q(x) + ax + b$ (1)

By the Remainder theorem: When P(x) is divided by $(x - c)$, the remainder is P(c).

$\therefore$ $P(3) = 0 + 3a + b = 3$
and $P(4) = 0 + 4a + b = 4$

$\therefore$ $3a + b = 3$ (2)
$4a + b = 4$ (3)

Subtracting (2) from (3): a = 1
Then from (2), b = 0.
So, from (1), when P(x) is divided by $(x - 3)(x - 4)$, the remainder is x.

Solution 5.1(5):

Let $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$

$\therefore$ $P'(x) = 4x^3 - 18x^2 + 24x - 10$
$P''(x) = 12x^2 - 36x + 24 = 12(x - 1)(x - 2)$

Since P(x) = 0 has a root of multiplicity 3, P'(x) has a zero of multiplicity 2 and P''(x) has a common zero with P(x).

Now $P''(x) = 12(x - 1)(x - 2)$
$P(1) = 1 - 6 + 12 - 10 + 3 = 0$
$P(2) = 16 - 48 + 48 - 40 + 3 \neq 0$

Hence x = 1 is the triple zero of P(x).

$\therefore$ $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$ (1)
$P(x) = (x - 1)^3(ax + b)$
$= (x^3 - 3x^2 + 3x - 1)(ax + b)$ (2)

Comparing the coefficients of x^3 and the constant terms of (1) and (2):

$a = 1, b = -3$

Hence the roots of P(x) = 0 are: x = 1, 1, 1, 3.

Solution 5.1(6): Let $P(z) = z^4 + 2z^3 + z^2 - 1 = 0$

$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a root of P(z) = 0

We use the theorem: If $z = (a + ib)$ is a root of P(z) = 0, then $z = (a - ib)$ is also a root of P(z) = 0. We form the quadratic

$z^2 - (z_1 + \bar{z}_1)z + z_1\bar{z}$ where $z_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore$ $z_1 + \bar{z}_1 = 2 \times (-\frac{1}{2}) = -1$

$z_1\bar{z}_1 = \frac{1}{4} + \frac{3}{4} = 1$

On dividing P(z) by $z^2 + z + 1$, we have:

$$\begin{array}{r|l} & z^2 + z - 1 \\ \hline z^2 + z + 1 & z^4 + 2z^3 + z^2 + 0\cdot z - 1 \\ & z^4 + z^3 + z^2 \\ \hline & z^3 \\ & z^3 + z^2 + z \\ \hline & -z^2 - z - 1 \\ & -z^2 - z - 1 \\ \hline & 0 \end{array}$$

$\therefore$ $P(z) = (z^2 + z + 1)(z^2 + z - 1)$

$\therefore$ $z^2 + z + 1 = 0$ or $z^2 + z - 1 = 0$

Hence the four roots are

$z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, \frac{-1 \pm \sqrt{5}}{2}$

Solution 5.1(7): The conjugate zeros of a polynomial with real coefficients occur in pairs.

$z = -3i$ and $z = 1 + 2i$ are the zeros of P(z).

$\therefore$ $z = 3i$ and $z = 1 - 2i$ are also the zeros of P(z).

The quadratic factor corresponding to the zeros 3i and –3i is $(z - 3i)[z - (-3i)] = z^2 + 9$

The quadratic factor corresponding to the other zeros $z = 1 \pm 2i$, is

$z^2 - (1 + 2i + 1 - 2i)z + (1 - 2i)(1 + 2i)$
$= z^2 - 2z + 5$

Hence the required polynomial P(z) of the lowest degree with real coefficients is:

$P(z) = (z^2 + 9)(z^2 - 2z + 5)$
$= z^4 - 2z^3 + 14z^2 - 18z + 45$

Solution 5.1(8):

$P(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$

$x = 1 + i$, $x^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

$x^4 = -4$

Now $P(x) = x^4 - 7x^2\cdot x + 18x^2 - 22x + 12$

Substituting $x = 1 + i$, $P(1 + i)$
$= -4 - 7\times 2i(1 + i) + 36i - 22(1 + i) + 12$
$= -4 - 14i + 14 + 36i - 22 - 22i + 12 = 0$

By the Factor theorem x = 1+i is a zero of P(x).
Now the complex zeros of P(x) with real coefficients occur in conjugate pairs. So, another zero of P(x) is x = 1 – i.
The quadratic factor corresponding to these two zeros is $x^2 - (1 + i + 1 - i)x + (1 + i)(1 - i)$
i.e. $x^2 - 2x + 2$
On dividing P(x) by $x^2 - 2x + 2$ we have:

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x^2 - 2x + 2 & x^4 - 7x^3 + 18x^2 - 22x + 12 \\ & x^4 - 2x^3 + 2x^2 \\ \hline & -5x^3 + 16x^2 - 22x \\ & -5x^3 + 10x^2 - 10x \\ \hline & 6x^2 - 12x + 12 \\ & 6x^2 - 12x + 12 \\ \hline & 0 \end{array}$$

$\therefore$ $P(x) = (x^2 - 2x + 2)(x^2 - 5x + 6)$
$= (x - 3)(x - 2)(x^2 - 2x + 2)$

Hence the zeros of P(x) are: $x = 2, 3, 1 \pm i$.

SOLUTIONS: EXERCISE 5.2

Solution 5.2(1): $x^3 - x^2 - 4x + 4 = 0$

We have:

$\alpha + \beta + \gamma = 1$ (1)
$\alpha\beta + \beta\gamma + \gamma\alpha = -4$ (2)
$\alpha\beta\gamma = -4$ (3)

(a) $\alpha^2 + \beta^2 + \gamma^2$
$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= 1 - 2\times(-4) = 9$

(b) We have:
$(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= \alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2 + 3\alpha\beta\gamma$

Substituting from (1), (2), (3) and rearranging:
$\Sigma\alpha^2\beta = 1 \times (-4) - 3 \times (-4) = 8$

(c) Substituting $x = \alpha, \beta, \gamma$ in the equation and then adding:

$\Sigma\alpha^3 - \Sigma\alpha^2 - 4\Sigma\alpha + 12 = 0$

$\therefore$ $\Sigma\alpha^3 = \Sigma\alpha^2 + 4\Sigma\alpha - 12$

Using the result of (a) and (1):

$\Sigma\alpha^3 = 9 + 4 - 12 = 1$

(d) We have: $x^3 = x^2 + 4x - 4$

Substituting $x = \alpha$ and then multiplying the resulting equation by α,

$\alpha^4 = \alpha^3 + 4\alpha^2 - 4\alpha$

Adding two similar expressions for β and γ to the last equation we have:

$\Sigma\alpha^4 = \Sigma\alpha^3 + 4\Sigma\alpha^2 - 4\Sigma\alpha$
$= 1 + 4\times 9 - 4\times 1 = 33$

(e) $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$

$= \dfrac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2}$

Now $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha$

$= 16 - 2 \times (-4) \times 1 = 24$

$\therefore \quad \Sigma\alpha^{-2} = \dfrac{24}{16} = \dfrac{3}{2}$

Solution 5.2(2): The roots of

$x^3 - 7x^2 + 14x - 8 = 0$ are in G. P.

Let the roots be: $\alpha = \dfrac{b}{r}$, $\beta = b$, $\gamma = br$,

being in G.P.

$\therefore \quad \alpha\beta\gamma = \dfrac{b}{r} \cdot b \cdot br = b^3$

Now $\alpha\beta\gamma$ = product of the roots = 8

$\therefore \quad b^3 = 8$, so $b = 2$

Hence one root of the equation is equal to 2.

We divide the left hand side by $(x - 2)$

$$
\begin{array}{r|l}
 & x^2 - 5x + 4 \\
x - 2 & x^3 - 7x^2 + 14x - 8 \\
 & x^3 - 2x^2 \\
 & \quad - 5x^2 + 14x \\
 & \quad - 5x^2 + 10x \\
 & \qquad 4x - 8 \\
 & \qquad 4x - 8 \\
 & \qquad\quad 0
\end{array}
$$

$\therefore \quad x^3 - 7x^2 + 14x - 8$

$= (x - 2)(x^2 - 5x + 4)$

$= (x - 2)(x - 1)(x - 4)$

Hence the roots of the given equation are:

1, 2, 4.

Solution 5.2(3): $6x^3 - 17x^2 - 5x + 6 = 0$

We have: $\alpha + \beta + \gamma = \dfrac{17}{6}$ (1)

$\alpha\beta\gamma$ = product of the roots $= -\dfrac{6}{6} = -1$

Now $\alpha\beta = -2$

$\therefore \quad -2\gamma = -1$, i.e. $\gamma = \dfrac{1}{2}$

We divide the equation by $(2x - 1)$

$$
\begin{array}{r|l}
 & 3x^2 - 7x - 6 \\
2x - 1 & 6x^3 - 17x^2 - 5x + 6 \\
 & 6x^3 - 3x^2 \\
 & \quad - 14x^2 - 5x \\
 & \quad - 14x^2 + 7x \\
 & \qquad -12x + 6 \\
 & \qquad -12x + 6 \\
 & \qquad\quad 0
\end{array}
$$

$\therefore \quad 6x^3 - 17x^2 - 5x + 6$

$= (2x - 1)(3x^2 - 7x - 6) = 0$

Now $3x^2 - 7x - 6 = (3x + 2)(x - 3)$

Hence the required roots are $x = \dfrac{1}{2}, 3, -\dfrac{2}{3}$

Solution 5.2(4): $x^3 + px^2 + qx + r = 0$ (1)

Let $y = x^2$, so that when $x = \alpha, \beta, \gamma$,

$y = \alpha^2, \beta^2, \gamma^2$.

We write Equation (1) so that the odd powers of x are on one side:

$x(x^2 + q) = -px^2 - r$

Squaring both sides

$x^2(x^4 + 2x^2q + q^2) = p^2x^4 + 2prx^2 + r^2$

Substituting $x^2 = y$

$y(y^2 + 2qy + q^2) = p^2y^2 + 2pry + r^2$

or $\quad y^3 + (2q - p^2)y^2 + (q^2 - 2rp)y - r^2 = 0$

is the required equation whose roots are

α^2, β^2 and γ^2.

$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 = -(2q - p^2) = p^2 - 2q$ (2)

and $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = q^2 - 2rp$ (3)

Solution 5.2(5):

$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ (1)

$x = \alpha$ and $x = \beta = -\alpha$ satisfy (1)

$\therefore \quad \alpha^4 - 2\alpha^3 + 4\alpha^2 + 6\alpha - 21 = 0$ (2)

and $\alpha^4 + 2\alpha^3 + 4\alpha^2 - 6\alpha - 21 = 0$ (3)

Adding (2) and (3): $2(\alpha^4 + 4\alpha^2 - 21) = 0$

$\therefore \quad (\alpha^2 + 7)(\alpha^2 - 3) = 0$

$\therefore \quad \alpha = \pm\sqrt{3}$ or $\alpha = \pm i\sqrt{7}$

$\alpha = \sqrt{3}$ and $\beta = -\sqrt{3}$ both satisfy Equation (1), but $\alpha = i\sqrt{7}$ does not satisfy (1).

We divide the equation by $x^2 - 3$, then:

$$
\begin{array}{r|l}
 & x^2 - 2x + 7 \\
x^2 - 3 & x^4 - 2x^3 + 4x^2 + 6x - 21 \\
 & x^4 \qquad - 3x^2 \\
 & \quad -2x^3 + 7x^2 + 6x \\
 & \quad -2x^3 \qquad + 6x \\
 & \qquad 7x^2 \qquad - 21 \\
 & \qquad 7x^2 \qquad - 21 \\
 & \qquad\quad 0
\end{array}
$$

$\therefore$ The other two roots of the equation are given by: $x^2 - 2x + 7 = 0$

$\therefore \quad x = \dfrac{2 \pm \sqrt{4 - 28}}{2} = 1 \pm i\sqrt{6}$

Hence the required roots are:

$x = \pm\sqrt{3}, \quad 1 \pm i\sqrt{6}$

Solution 5.2(6): $x^3 - px^2 + qx - r = 0$

Let the roots be $a - d$, a, $a + d$, so that they are in A. P. $\therefore \quad a - d + a + a + d = p$

$3a = p$ or $a = \dfrac{p}{3}$ (1)

Continued

$x = a = \dfrac{p}{3}$ satisfies the equation.

$\therefore \quad \dfrac{p^3}{27} - p \cdot \dfrac{p^2}{9} + q \cdot \dfrac{p}{3} - r = 0$

This simplifies to

$2p^3 - 9pq + 27r = 0$ (2)

which is the required condition for the equation to have roots in A. P.

To solve the equation

$x^3 - 12x^2 + 39x - 28 = 0$ (3)

we note that $p = 12$, $q = 39$, $r = 28$

Substituting in L. H. S. of (2):

$2 \times 1728 - 9 \times 12 \times 39 + 27 \times 28$,

this is equal to zero, so Equation (3) has roots which are in A. P. From (1) one root is

$x = \dfrac{p}{3} = \dfrac{12}{3} = 4$

We divide the L.H.S. of the equation by $(x - 4)$

$$
\begin{array}{r|l}
 & x^2 - 8x + 7 \\
x - 4 & x^3 - 12x^2 + 39x - 28 \\
 & x^3 - 4x^2 \\
 & \quad - 8x^2 + 39x \\
 & \quad -8x^2 + 32x \\
 & \qquad 7x - 28 \\
 & \qquad 7x - 28 \\
 & \qquad\quad 0
\end{array}
$$

Now the roots of $x^2 - 8x + 7 = 0$ are given by

$(x - 7)(x - 1) = 0$, i.e. $x = 1$ or 7.

Hence the required roots are 1, 4, 7 which are clearly in A. P.

Solution 5.2(7): Let $P(x) = 2x^3 - 9x^2 + 12x + k$

then $\quad \dfrac{dP}{dx} = 6x^2 - 18x + 12$

$= 6(x - 2)(x - 1)$

Since $P(x)$ has two equal zeros, $\dfrac{dP}{dx}$ and $P(x)$ must have a common root.

Now $\dfrac{dP}{dx} = 0$ gives $x = 1$ or 2.

If $x = 1$ satisfies $P(x) = 0$, then

$2 - 9 + 12 + k = 0$, giving $k = -5$.

If $x = 2$ satisfies $P(x) = 0$, then

$16 - 36 + 24 + k = 0$, giving $k = -4$

When $k = -5$, we write

$P(x) = (x - 1)^2(ax + b)$

$= (x^2 - 2x + 1)(ax + b)$

Comparing with $P(x) = 2x^3 - 9x^2 + 12x - 5$

$a = 2$, $b = -5$

Hence the three roots for $k = -5$ are

$x = 1, 1, \dfrac{5}{2}$

Again when $k = -4$, the equation

$P(x) = 2x^3 - 9x^2 + 12x - 4$ (1)

$= (x - 2)^2(mx + n)$

$= (x^2 - 4x + 4)(mx + n)$ (2)

Comparing the coefficients of x^3 and the constant terms in (1) and (2),

$m = 2$, $4n = -4$ or $n = -1$

For $k = -4$, the roots are $x = 2, 2, \dfrac{1}{2}$

Hence the two values of k are -5 and -4 and the corresponding roots are:

$\left(1, 1, \dfrac{5}{2}\right)$ or $\left(2, 2, \dfrac{1}{2}\right)$

Solution 5.2(8):

$x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ (1)

We have $\alpha + \beta = \gamma + \delta$ (2)

Now $\alpha + \beta + \gamma + \delta = 8$

Using (2), $\alpha + \beta = \gamma + \delta = 4$ (3)

Let $\alpha\beta = m$ and $\gamma\delta = n$, then

$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - 4x + m$

and $\quad x^2 - (\gamma + \delta)x + \gamma\delta = x^2 - 4x + n$

Hence

$x^4 - 8x^3 + 21x^2 - 20x + 5$

$= (x^2 - 4x + m)(x^2 - 4x + n)$

Comparing the coefficients of like powers of x, we have: $-4(m + n) = -20$ and $mn = 5$

$\therefore \quad m + n = 5$ and $mn = 5$

$\therefore \quad$ m and n are the roots of $y^2 - 5y + 5 = 0$

$\therefore \quad y = \dfrac{5 \pm \sqrt{5}}{2}$

Hence the two quadratics are:

$x^2 - 4x + \left(\dfrac{5 + \sqrt{5}}{2}\right) = 0$ (4)

and $x^2 - 4x + \left(\dfrac{5 - \sqrt{5}}{2}\right) = 0$ (5)

Solving (4) and (5) the required roots are:

$\dfrac{4 \pm \sqrt{16 - 2(5 + \sqrt{5})}}{2}, \dfrac{4 \pm \sqrt{16 - 2(5 - \sqrt{5})}}{2}$

i.e. $\dfrac{4 \pm \sqrt{6 - 2\sqrt{5}}}{2}, \dfrac{4 \pm \sqrt{6 + 2\sqrt{5}}}{2}$

Now $6 \pm 2\sqrt{5} = (\sqrt{5} \pm 1)^2$, hence the required roots are: $\dfrac{3 \pm \sqrt{5}}{2}, \dfrac{5 \pm \sqrt{5}}{2}$

Solution 5.2(9): $x^4 + px^3 + qx^2 + rx + s = 0$ (1)

Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$

$\therefore \quad$ sum of the roots is

$a - 3d + a - d + a + d + a + 3d = -p$

$\therefore \quad a = -\dfrac{p}{4}$ (2)

$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

Substituting for α, β, γ and δ,

$\Sigma\alpha\beta = 6a^2 - 10d^2 = q$

Continued

Substituting for α, β, γ and δ,

$\Sigma\alpha\beta = 6a^2 - 10d^2 = q$ Continued

Using $a = -\frac{p}{4}$, $6\cdot\frac{p^2}{16} - 10d^2 = q$

$\therefore \quad d^2 = \dfrac{3p^2 - 8q}{80}$ (3)

$\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

Substituting for α, β, γ and δ,

$\Sigma\alpha\beta\gamma = 4a(a^2 - 5d^2) = -r$ (4)

$\alpha\beta\gamma\delta = (a^2 - d^2)(a^2 - 9d^2) = s$ (5)

The required conditions are obtained by putting the expressions for a and d^2 in (4) and (5).

These are: $p^3 - 4pq + 8r = 0$ (6)

and $(36q - 11p^2)(4q + p^2) = 1600s$

For the equation:

$x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$

$\Sigma\alpha = 4a = -2$, giving $a = -\frac{1}{2}$

$\Sigma\alpha\beta = 6a^2 - 10d^2 = -21$

$\therefore \quad \frac{6}{4} - 10d^2 = -21$

$\therefore \quad d^2 = \frac{9}{4}$, giving $d = \pm\frac{3}{2}$

Hence the roots are: $-5, -2, 1, 4$, which are clearly in A. P.

Solution 5.2(10): $x^3 + 3x + 9 = 0$ (1)

$\therefore \quad x^3 = -3(x + 3)$ (2)

Substituting $x = \alpha$

$\alpha^3 = -3(\alpha + 3)$

Cubing both sides:

$\alpha^9 = -27(\alpha^3 + 9\alpha^2 + 27\alpha + 27)$

Adding to this the similar expressions for β and γ

$\Sigma\alpha^9 = -27(\Sigma\alpha^3 + 9\Sigma\alpha^2 + 27\Sigma\alpha + 81)$

Now from (2)

$\Sigma\alpha^3 = -3(\Sigma\alpha + 9)$

From (1), $\Sigma\alpha = 0$, hence $\Sigma\alpha^3 = -27$ (3)

$\Sigma\alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2\Sigma\alpha\beta = 0 - 2 \times 3$

$= -6$ (4)

$\therefore \quad \Sigma\alpha^9 = -27[-27 + 9\times(-6) + 0 + 81]$

$= -27 \times 0 = 0$

$\therefore \quad \alpha^9 + \beta^9 + \gamma^9 = 0$

SOLUTIONS: EXERCISE 5.3

Solution 5.3(1): $z^2 + 2\sqrt{2}\,iz + 2\sqrt{3}\,i = 0$

Using the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\sqrt{2}i \pm \sqrt{-8 - 8\sqrt{3}i}}{2}$$

$$= -\sqrt{2}i \pm \sqrt{-2 - 2\sqrt{3}i}$$

Now, $-2 - 2\sqrt{3}i = (1 - \sqrt{3}i)^2$

$\therefore \quad z = -\sqrt{2}i \pm (1 - \sqrt{3}i)$

$= 1 - (\sqrt{2} + \sqrt{3})i, \ -1 + (\sqrt{3} - \sqrt{2})i$

Solution 5.3(2):

$x^4 + mx^3 + 13x^2 + nx - 36 = 0$ (1)

Denoting the L.H.S. of (1) by P(x), and using the given data that $x = 3$ is a double root of (1), P(x) and P'(x) have a common zero $x = 3$.

Now $P'(x) = 4x^3 + 3mx^2 + 26x + n$ (2)

$x = 3$ satisfies (1) and (2)

$\therefore \quad 81 + 27m + 117 + 3n - 36 = 0$

and $108 + 27m + 78 + n = 0$

These simplify to

$27m + n = -186$ (3)

$27m + 3n = -162$ (4)

Subtracting (3) from (4):

$2n = 24$ giving $n = 12$

and from (3) $m = -\frac{198}{27} = -\frac{22}{3}$

Solution 5.3(3): We have $x^4 - 3x^2 + 2x - 1$

$= Ax(x - 1)(x - 2)(x - 3)$

$+ Bx(x - 1)(x - 2) + Cx(x - 1)$

$+ Dx + E$ (1)

(1) is an identity, so we may substitute any suitable value of x in (1). We select those values of x, which make most of the R.H.S vanish. Put $x = 0, 1, 2, 3$ in succession, then

when $x = 0$, $-1 = E$ or $E = -1$, (2)

when $x = 1$, $1 - 3 + 2 - 1 = D + E$

$\therefore \quad D + E = -1$, so $D = 0$, (3)

when $x = 2$, $16 - 12 + 4 - 1 = 2C + 2D + E$

$2C + 2D + E = 7$, $2C + 0 - 1 = 7$

$\therefore \quad C = 4$, (4)

when $x = 3$, $81 - 27 + 6 - 1$

$= 6B + 6C + 3D + E$

$\therefore \quad 6B + 6C + 3D + E = 59$

$\therefore \quad 6B + 24 + 0 - 1 = 59$

$\therefore \quad B = 6$ (5)

It is easier to compare the coefficients of x^4 on both sides of (1), then substitute any value of x.

$\therefore \quad A = 1$ (6)

Hence, $A = 1$, $B = 6$, $C = 4$, $D = 0$, $E = -1$

You can also find the values of A, B, C, D, E by comparing like powers in the two sides of (1).

Solution 5.3(4): Since P(x) has a triple zero $x = c$, we can write $P(x) = (x - c)^3Q(x)$, where $Q(c) \neq 0$.

$\therefore \quad P'(x) = 3(x - c)^2.Q(x) + (x - c)^3.Q'(x)$

$= (x - c)^2[3Q(x) + (x - c)\cdot Q'(x)]$

Hence P'(x) has a double zero $x = c$.

For simplicity, we let

$R(x) = 3Q(x) + (x - c)\cdot Q'(x)$

$P'(x) = (x - c)^2R(x)$ Continued

$\therefore \quad P''(x) = 2(x - c)\cdot R(x) + (x - c)^2\cdot R'(x)$

$= (x - c)[2R(x) + (x - c)\cdot R'(x)]$

Hence $x = c$ is a zero of P''(x).

Thus $x = c$ is a zero of P(x), P'(x) and P''(x) if P(x) has a triple zero.

Solution 5.3(5): When $x^3 + px + q$ is divided by $(x - 2)(x + 3)$, the remainder is $2x + 1$.

$\therefore \quad P(x) = x^3 + px + q = (x - 2)(x + 3)Q(x) + 2x + 1$ (1)

(1) is an identity, so when $x = 2$,

$P(2) = 8 + 2p + q = 5$

$x = -3$, $P(-3) = -27 - 3p + q = -5$

These simplify to:

$2p + q = -3$

and $-3p + q = 22$

Subtracting $5p = -25$, gives $p = -5$

Hence $-10 + q = -3$ $\therefore \quad q = 7$

Solution 5.3(6): $P(x) = x^5 - px^2 + q$ (1)

If P(x) has a multiple zero, then P(x) and P'(x) have a common zero, say $x = c$.

$\therefore \quad c^5 - pc^2 + q = 0$ (2)

$5c^4 - 2pc = 0$ (3)

$c \neq 0$, so from (3) $p = \frac{5c^3}{2}$ (4)

Substituting $p = \frac{5c^3}{2}$ in (2):

$c^5 - \frac{5}{2}c^5 + q = 0$

$\therefore \quad c^5 = \frac{2}{3}q$ (5)

From (4), $c^3 = \frac{2p}{5}$

$\therefore \quad c^{15} = \left(\frac{2p}{5}\right)^5$

From (5), $c^{15} = \left(\frac{2q}{3}\right)^3$

$\therefore \quad \left(\frac{2p}{5}\right)^5 = \left(\frac{2q}{3}\right)^3$

Simplifying, $108p^5 = 3125q^3$,

which is the required condition for P(x) to have a multiple zero.

Solution 5.3(7):

Let $P(x) = x^n + mx - b = 0$ (1)

If P(x) has a multiple zero, then P(x) and P'(x) have a common zero, say $x = c$.

$\therefore \quad c^n + mc - b = 0$ (2)

$nc^{n-1} + m = 0$ or $c^{n-1} = -\frac{m}{n}$ (3)

Multiplying by c, $c^n = -\frac{mc}{n}$

Substituting in (2)

$-\frac{mc}{n} + mc - b = 0$

$\therefore \quad c = \frac{bn}{m(n - 1)}$

$\therefore \quad c^{n-1} = \left[\frac{bn}{m(n - 1)}\right]^{n-1}$ (4)

From (4) and (3):

$\dfrac{b^{n-1}\cdot n^{n-1}}{m^{n-1}\cdot(n - 1)^{n-1}} = -\dfrac{m}{n}$

This simplifies to $\left(\frac{m}{n}\right)^n = -\dfrac{b^{n-1}}{(n - 1)^{n-1}}$

or $\left(\frac{m}{n}\right)^n + \left(\frac{b}{n - 1}\right)^{n-1} = 0$,

which is the required condition for

$x^n + mx - b = 0$ to have a multiple root.

Solution 5.3(8): $x^3 - x - 1 = 0$ (1)

α, β, γ are the roots of Equation (1).

To find the equation whose roots are of the form $\frac{1 + \alpha}{1 - \alpha}$, let $y = \frac{1 + x}{1 - x}$, so that when

$x = \alpha$, $y = \frac{1 + \alpha}{1 - \alpha}$

From $y = \frac{1 + x}{1 - x}$, $y - xy = 1 + x$

$\therefore \quad x = \frac{y - 1}{y + 1}$

Substituting in (1):

$\left(\frac{y - 1}{y + 1}\right)^3 - \left(\frac{y - 1}{y + 1}\right) - 1 = 0$

Multiplying by $(y + 1)^3$

$(y - 1)^3 - (y - 1)(y + 1)^2 - (y + 1)^3 = 0$

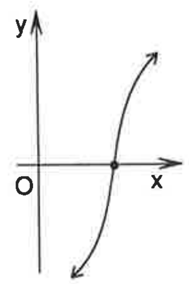
This simplifies to:

$y^3 + 7y^2 - y + 1 = 0.$

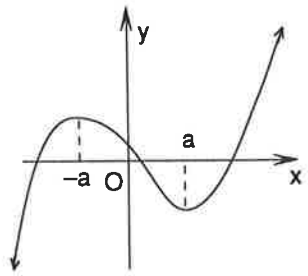
We may replace y by x; then the required equation is

$x^3 + 7x^2 - x + 1 = 0.$

Solution 5.3(9):



(Fig. 1



(Fig. 2)

(a) $P(x) = x^5 - 5px + q$ (1)

$P'(x) = 5x^4 - 5p = 5(x^4 - p)$ (2)

If $p < 0$, P'(x) can never vanish, hence P(x) has no turning points.

$P(x) \to \infty$ as $x \to \infty$ and

$P(x) \to -\infty$ as $x \to -\infty$

Continued

So P(x) has at least one real root in the interval $(-\infty, \infty)$. Looking at Fig. 1 and the above information, we can state that if $p < 0$, then P(x) has only one real zero.

(b) The stationary points of P(x) are given by

$P'(x) = 0$

$\therefore \quad x^4 - p = 0$

This equation has real roots if $p \geq 0$.

Let $p = a^4$, then

$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2) = 0$

$\therefore$ There are only two turning points given by $x = \pm a = \pm p^{1/4}$.

For 3 real roots, these turning points must be on opposite sides of the x-axis.

Now $P(x) = x(x^4 - 5p) + q$

$\therefore \quad P(a) = a(a^4 - 5a^4) + q = q - 4a^5$

Similarly $P(-a) = q + 4a^5$

Using the fact that P(a) and P(–a) are of opposite signs, $P(a)\cdot P(-a) < 0$

$\therefore \quad (q - 4a^5)(q + 4a^5) < 0$

$q^2 - 16a^{10} < 0$

$\therefore \quad q^2 < 16a^{10} \qquad q^4 < 256a^{20}$

Substituting $a^4 = p$, $q^4 < 256p^5$,

which is the required condition for P(x) to have 3 distinct real roots.

Solution 5.3(10):

$P(x) = x^3 - x^2 - 5x - 1 = 0 \qquad (1)$

$P'(x) = 3x^2 - 2x - 5 = (3x - 5)(x + 1)$

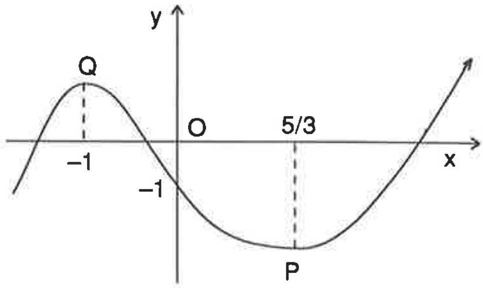
The turning points of P(x) are given by

$(3x - 5)(x + 1) = 0$

$\therefore \quad x = \frac{5}{3}$ or $x = -1$

Now $P(-1) = -1 - 1 + 5 - 1 = 2$

$P(5/3) = -\frac{202}{27}, \qquad P(0) = -1$



The two turning points

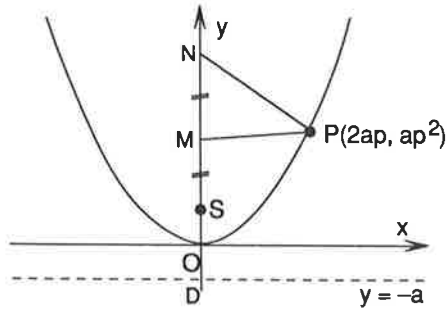
$P(\frac{5}{3}, -\frac{202}{27})$ and $Q(-1, 2)$

are on opposite sides of the x-axis, so the curve $y = P(x)$ intersects the x-axis in 3 distinct points and hence P(x) has 3 distinct zeros.

SOLUTIONS OF PROBLEMS: CHAPTER 6

SOLUTIONS: EXERCISE 6.1

Solution 6.1(1):



The equation of the normal; at $P(2ap, ap^3)$ is

$x + py = 2ap + ap^3 \qquad (1)$

Putting $x = 0$, $y = 2a + ap^2$

$\therefore$ N is $(0, 2a + ap^2)$, S is $(0,a)$

M is the mid-point of NS.

$\therefore$ M is $\left(0, \frac{3a + ap^2}{2}\right)$

$P(2ap, ap^2)$, $D(0, -a)$

$\therefore \quad DM = DO + OM = a + \frac{3a + ap^2}{2}$

$DM = \frac{5a + ap^2}{2} \qquad (2)$

$MP^2 = (2ap)^2 + \left(ap^2 - \frac{3a + ap^2}{2}\right)^2$

$= 4a^2p^2 + \left(\frac{ap^2 - 3a}{2}\right)^2$

$= \frac{10a^2p^2 + a^2p^4 + 9a^2}{4} \qquad (3)$

Hence, from (2) and (3):

$DM^2 - MP^2 = \frac{25a^2 + 10a^2p^2 + a^2p^4}{4} - \frac{10a^2p^2 + a^2p^4 + 9a^2}{4} = 4a^2$

$\therefore \quad DM^2 - MP^2 = 4a^2$.

Solution 6.1(2):(a) Let $P(x_1, y_1)$ be the point $(2ap, ap^2)$ and $Q(x_2, y_2)$ be the point $(2aq, aq^2)$.

The equation of the chord PQ is:

$y - ap^2 = \frac{ap^2 - aq^2}{2ap - 2aq}(x - 2ap)$

This simplifies to:

$y = \frac{1}{2}(p + q)x - apq$

If this chord passes through the focus S(0,a), then: $\quad a = -apq$

Multiplying by 4a

Continued