SELF-TESTING EXERCISE 5.3 (For complete solutions see page 250.)

1. Solve $z^2 + 2\sqrt{2}\,iz + 2\sqrt{3}\,i = 0$, giving your answer in the form $A + iB$.
2. Find m and n such that $x = 3$ is a double root of $x^4 + mx^3 + 13x^2 + nx - 36 = 0$.
3. Express $x^4 - 3x^2 + 2x - 1$ in the form

   $$Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + Dx + E.$$

   You are required to find A, B, C, D, E.
4. Show that if $P(x)$ has a triple root $x = c$, then $x = c$ is a root of $P'(x)$ and $P''(x)$, where $P(x)$ is a polynomial of degree $n \geq 3$.
5. The remainder when $x^3 + px + q$ is divided by $(x-2)(x+3)$ is $2x + 1$. Find p and q.
6. Prove that $P(x) = x^5 - px^2 + q$ has a multiple root if $108\,p^5 = 3125q^3$.
7. Show that $x^n + mx - b = 0$ has a multiple root provided $\left(\frac{m}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$.
8. If α, β, γ are the roots of $x^3 - x - 1 = 0$, find the equation whose roots are:

   $$\frac{1+\alpha}{1-\alpha},\ \frac{1+\beta}{1-\beta} \text{ and } \frac{1+\gamma}{1-\gamma}.$$
9. A polynomial $P(x)$ is given by $P(x) = x^5 - 5px + q$, where p and q are real.

   (a) By considering the turning points, prove that if $p < 0$, $P(x) = 0$ has only one real root.

   (b) Prove that $P(x) = 0$ has 3 distinct real roots if $q^4 < 256\,p^5$.
10. By considering the turning points of the polynomial $P(x) = x^3 - x^2 - 5x - 1$, show that the equation $P(x) = 0$ has 3 distinct real roots.

SELF-TESTING EXERCISE 5.1 (For complete solutions see page 246.)

1. Express each of the following polynomials as a product of linear factors over the complex field C: (a) $16x^4 - 1$, (b) $4x^2 + x + 3$, (c) $x^4 + x^2 + 1$
2. If $P(x) = 2x^4 + x^2 - i$, find the remainder when $P(x)$ is divided by $x - (1 - 2i)$.
3. Determine b and c such that $x^4 + bx^3 + cx^2 - x + 2$ is divisible by $x^2 - 1$.
4. When $P(x)$ is divided by $x - 3$ and $x - 4$ the remainders are 3 and 4 respectively. Find the remainder when $P(x)$ is divided by $(x - 3)(x - 4)$.
5. Solve $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ given that the equation has a root of multiplicity 3.
6. If $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a root of $z^4 + 2z^3 + z^2 - 1 = 0$, find all the other roots.
7. Find a polynomial $P(x)$ of lowest degree with real coefficients having $-3i$ and $1 + 2i$ as zeros. (Hint: conjugate roots).
8. Prove that $x = 1 + i$ is a zero of $P(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$ and hence find all the other roots. [Hint: Show $P(1 + i) = 0$].

SELF-TESTING EXERCISE 5.2 (For complete solutions see page 247.)

1. If α, β, γ are the roots of $x^3 - x^2 - 4x + 4 = 0$, evaluate:
   (a) $\alpha^2 + \beta^2 + \gamma^2$, (b) $\Sigma\alpha^2\beta$, (c) $\Sigma\alpha^3$, (d) $\Sigma\alpha^4$, (e) $\Sigma\alpha^{-2}$
2. Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$, given that the roots of the equation are in geometric progression.
3. Solve $6x^3 - 17x^2 - 5x + 6 = 0$, given that $\alpha\beta = -2$, where α, β, γ are the roots of the equation.
4. α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$, find the equation whose roots are α^2, β^2 and γ^2 and hence evaluate: $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta^2$.
5. $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$. If $\alpha + \beta = 0$, solve the equation completely.
6. Prove that the condition for the roots of $x^3 - px^2 + qx - r = 0$ to be in arithmetic progression is $2p^3 - 9pq + 27r = 0$ and hence or otherwise solve $x^3 - 12x^2 + 39x - 28 = 0$.
7. The cubic $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots. Find k and solve the equation. (Two solutions.)
8. α, β, γ and δ are the roots of $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$, such that $\alpha + \beta = \gamma + \delta$. Solve the equation.
9. Prove that the conditions for the equation $x^4 + px^3 + qx^2 + rx + s = 0$ to have roots in arithmetic progression are $p^3 - 4pq + 8r = 0$ and $(36q - 11p^2)(4q + p^2) = 1600s$. Hence solve $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$.
10. Show that $\alpha^9 + \beta^9 + \gamma^9 = 0$, where α, β, γ are the roots of $x^3 + 3x + 9 = 0$. [Hint: write $x^3 = -3(x + 3)$ and cube both sides.]