Example: $P(x)$ is a polynomial with integer coefficients.

(a) Show that if α is an integral zero of $P(x)$, then α is a divisor of the constant term.

Solution:

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

$$P(\alpha) = 0$$

$$\therefore 0 = a_n \alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0$$

$$\therefore -a_0 = \alpha(a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \dots + a_1)$$

But $P(x)$ has integer coefficients and α is integral

$\therefore -a_0 = \alpha \times k$ where $k \in \mathbb{Z}$

$\therefore \dfrac{-a_0}{\alpha} = k$ and hence α is a divisor of a_0.

Note: $k = a_n\alpha^{n-1} + a_{n-1}\alpha^{n-2} + \dots + a_1$

(b) Show that if $\beta = \frac{p}{q}$ is a rational zero of $P(x)$, where p and q have no common factor, then p is a divisor of the constant term and q is a divisor of the leading coefficient.

Solution:

$$P\left(\frac{p}{q}\right) = 0$$

$$\therefore 0 = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_1\frac{p}{q} + a_0$$

$$0 = a_n\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \dots + a_1\frac{p}{q} + a_0$$

Multiply both sides by q^n

$$\therefore 0 = a_np^n + a_{n-1}p^{n-1}q + \dots + a_1pq^{n-1} + a_0q^n$$

$$-a_0q^n = p\left[a_np^{n-1} + a_{n-1}p^{n-2}q + \dots + a_1q^{n-1}\right]$$

But $P(x)$ has integer coefficients and $p, q \in \mathbb{Z}$.

$\therefore -a_0q^n = p \times k$ where $k \in \mathbb{Z}$

and $k = a_np^{n-1} + a_{n-1}p^{n-2}q + \dots + a_1q^{n-1}$

Hence $-\frac{a_0q^n}{p} = k$

But p, q have no common factors

$\therefore$ p and q^n have no common factors

$\therefore$ p is not a divisor of q^n

$\boxed{\text{Hence } p \text{ is a divisor of } a_0.}$

$$-\frac{a_0 q^n}{p} = k$$

RTP: q is a divisor of a_n

Earlier in the solution we showed that

$$0 = a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n$$

$$-a_n p^n = q\left[a_{n-1} p^{n-1} + \dots + a_1 p q^n + a_0 q^{n-1}\right]$$

But $P(x)$ has integer coefficients and $p, q \in \mathbb{Z}$.

$\therefore$ $a_{n-1} p^{n-1} + \dots + a_1 p q^n + a_0 q^{n-1} = k,$

where $k \in \mathbb{Z}$

$$\therefore -a_n p^n = q \times k$$

$$-\frac{a_n p^n}{q} = k$$

But p, q have no common factors

$\therefore$ q and p^n have no common factors

$\therefore$ q is not a divisor of p^n.

Hence, q is a divisor of a_n.

(c) Show that $4x^3+2x^2-14x+3$ has exactly one rational zero and factorise the given polynomial over $\mathbb{Q}$.

from (b)

<u>Solution:</u>

Let $P(x) = 4x^3+2x^2-14x+3$

All rational zeros have the form $\frac{p}{q}$ where p and q are integer divisors of 3 & 4 respectively.

possible values for p and q are

$p = \pm 1, \pm 3$; $q = \pm 1, \pm 4, \pm 2$

Through trial & error we try the possible rational zeros of

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{3}{2}, \pm \frac{3}{4}$

Of these, only $\frac{3}{2}$ satisfies $P(x)=0$

$\therefore x = \frac{3}{2} \Rightarrow x - \frac{3}{2} = 0 \Rightarrow 2x-3=0$

$\therefore (2x-3)$ is a factor of $P(x)$

By inspection or polynomial long division, $P(x) = (2x-3)(2x^2+4x-1)$ and these are irreducible factors over $\mathbb{Q}$.

Multiple Zero's of a Polynomial

(1) If α is a zero of multiplicity r of $P(x)$, $r>1$, then α is a zero of multiplicity $r-1$ of $P'(x)$

e.g. $P(x) = (x-a)^r Q(x)$

$$P'(x) = r(x-a)^{r-1} Q(x) + (x-a)^r Q'(x)$$

$$= (x-a)^{r-1}\left[r Q(x) + (x-a) Q'(x)\right]$$

$$\boxed{P'(x) = (x-a)^{r-1} T(x)}$$

Why is $r>1$?

Consider $P(x) = (x-a)^1 Q(x)$

$$P'(x) = 1 \times Q(x) + (x-a) Q'(x)$$

$\therefore$ $(x-a)$ is not a factor of $P'(x)$ and $\therefore$ $x=a$ is not a zero of $P(x)$

Multiple zeros of a Polynomial

(2) Multiplicity of a zero

If $P(x)$ and $P'(x)$ have a common zero α, then α is a multiple zero of $P(x)$. Further, if $P(\alpha) = P'(\alpha) = P''(\alpha) = \ldots = P^{r-1}(\alpha) = 0$ and $P^{r}(\alpha) \neq 0$, then α is a zero of multiplicity r of $P(x)$.

To understand this, consider $P(x) = (x-2)^4 Q(x)$

$$P'(x) = (x-2)^3 R(x) \rightarrow P'(2) = 0$$

$$P''(x) = (x-2)^2 S(x) \rightarrow P''(2) = 0$$

$$P'''(x) = (x-2)^1 T(x) \rightarrow P'''(2) = 0$$

$$P^{iv}(x) = U(x) \rightarrow P^{iv}(2) = U(2) \neq 0$$

Thus, a factor of multiplicity r can be differentiated $r-1$ times before it 'dissapears'.

$P(x)$ can be differentiated more than r times in many cases. since $P(x) = (x-\alpha)^r Q(x)$

However, the $P^{r-1}(\alpha) = 0$ is simply saying that the factor $x-\alpha$ with multiplicity r can be differentiated at most $r-1$ times; $P'(\alpha) = P''(\alpha) \ldots = P^{r-1}(\alpha) = 0$

The $r-1$ refers to the number of dashes for the derivative.

Polynomial Proofs

If $P(x) = (x-\alpha)^r Q(x)$ where $Q(x)$ is a polynomial such that $Q(\alpha) \neq 0$.

$\therefore$ $P'(x) = (x-\alpha)^{r-1} Q_1(x)$ where $Q_1(\alpha) \neq 0$

Hence, if α is a zero of multiplicity r of $P(x)$, then α is a zero of multiplicity $r-1$ of $P'(x)$.

Conversely, let $P(x)$ and $P'(x)$ have a common zero α.

Then $P(x) = (x-\alpha)Q(x)$ and $P'(x) = Q(x) + (x-\alpha)Q'(x)$

But $P'(\alpha) = 0 \Rightarrow Q(\alpha) = 0$

$\therefore$ $(x-\alpha)$ is a factor of $Q(x)$

Hence $(x-\alpha)^2$ is a factor of $P(x)$ and α is a multiple zero of $P(x)$.

$\therefore$ The multiplicity of α as a zero of $P(x)$ is one more than the multiplicity of α as a zero of $P'(x)$.

Now, let α be a zero of $P(x)$, such that $P(\alpha) = P'(\alpha) = P''(\alpha) = \dots = P^{r-1}(\alpha) = 0$, $P^r(\alpha) \neq 0$

Then α is a single zero of $P^{r-1}(x)$ and a common zero of $P^{r-2}(x)$

Hence α is a zero of multiplicity 2 of $P^{r-2}(x)$.

$\therefore$ Each step backwards along the chain of derivatives increases the multiplicity of the zero α by 1, so that as a zero of $P(x)$, α has multiplicity r.

3/ Example:

(a) Show that $P(x) = x^4 - 2x^3 + 2x - 1$ has a multiple zero.

Solution:

$P(1) = 1 - 2 + 2 - 1$
$= 0$;
$P'(x) = 4x^3 - 6x^2 + 2$
$P'(1) = 4 - 6 + 2$
$= 0$

$\therefore P(1) = P'(1) = 0$
$\Rightarrow$ multiple zero at $x = 1$

$P''(x) = 12x^2 - 12x$

$P\ (1) = 12 - 12$
$= 0$

$P'''(x) = 24x - 12$

$P'''(1) = 24 - 12$
$= 12$
$\neq 0$

$\therefore x = 1$ is a zero of multiplicity 3 since $P''(1) = 0$ and $P'''(1) \neq 0$.