For $ax^3+bx^2+cx+d=0$, the sum of the roots $= -\frac{b}{a}$

$\therefore \alpha+\beta+\gamma = -\frac{b}{a}$

For $8x^3-6x-1=0 \Rightarrow b=0$

$\therefore \cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = -\frac{0}{8}$

$\therefore \cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$

Example: (a) Use De Moivre's Theorem to express $\tan 5\theta$ in terms of $\tan\theta$

(b) Hence show that $x^4-10x^2+5=0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$.

(c) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5}=5$

(d) By solving $x^4-10x^2+5=0$ another way, find the value of $\tan\frac{\pi}{5}$ as a surd.

Solution:

(a) $\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$$= \frac{{}^5C_1 \sin\theta\cos^4\theta - {}^5C_3 \sin^3\theta\cos^2\theta + {}^5C_5 \sin^5\theta}{\cos^5\theta - {}^5C_2 \sin^2\theta\cos^3\theta + {}^5C_4 \sin^4\theta\cos\theta}$$

Using DeMoivre's Theorem / Binomial Theorem.

Divide numerator & denominator by $\cos^5\theta$

$$\therefore \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(b) Let $x = \tan\theta$

$$\therefore \tan 5\theta = \frac{5x - 10x^3 + x^5}{1 - 10x^2 + 5x^4}$$

$$\tan 5\theta = \frac{x(x^4 - 10x^2 + 5)}{5x^4 - 10x^2 + 1}$$

Factorising

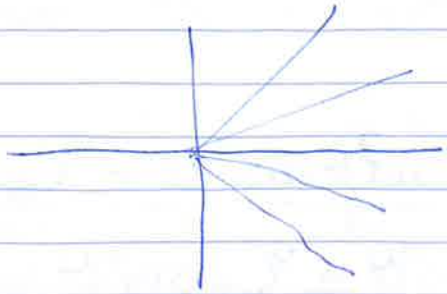
$\therefore x(x^4 - 10x^2 + 5) = 0 \Leftrightarrow \tan 5\theta = 0$

$\therefore$ The roots of $x^4 - 10x^2 + 5 = 0$ are the non zero values of $\tan\theta$, where θ is a solution of $\tan 5\theta = 0$.

$\therefore 5\theta = (0 + n\pi)$

$\theta = \dfrac{n\pi}{5} \qquad n \in \mathbb{Z}$

$\therefore \theta = \overset{\text{non-zero}}{\cancel{0}}, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}, \ldots$



Of these values of θ, there are 4 distinct non zero values of $\tan\theta$!

They are $\tan\left(\pm\frac{\pi}{5}\right)$; $\tan\left(\pm\frac{2\pi}{5}\right)$

$\tan\frac{\pi}{5}, -\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, -\tan\frac{2\pi}{5}$

$\therefore$ These are the roots of

$x^4 - 10x^2 + 5 = 0$

(c) RTP: $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$

A

$\angle\frac{\pi}{5}$ $-\tan\frac{\pi}{5} = \tan\frac{4\pi}{5}$

C $-\tan\frac{3\pi}{5} = \tan\frac{3\pi}{5}$

$\angle\frac{2\pi}{5}$

$\Rightarrow$ Product of roots $= \frac{e}{a}$

$\Rightarrow \tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \tan\frac{3\pi}{5} \times \tan\frac{4\pi}{5} = \frac{5}{1}$

QED.

(d) $x^4 - 10x^2 + 5 = 0$ Let $u = x^2$

$\Leftrightarrow u^2 - 10u + 5 = 0$

$$u = \frac{10 \pm \sqrt{80}}{2}$$

$$= \frac{10 \pm 4\sqrt{5}}{2}$$

$\therefore x^2 = 5 \pm 2\sqrt{5}$

But $x = \pm \tan\frac{\pi}{5} \Rightarrow x^2 = \tan^2\frac{\pi}{5}$

$x = \pm \tan\frac{2\pi}{5} \Rightarrow x^2 = \tan^2\frac{2\pi}{5}$

$\tan^2\frac{\pi}{5} = 5 \pm 2\sqrt{5}$ or $\tan^2\frac{2\pi}{5} = 5 \pm 2\sqrt{5}$

But $0 < \tan\frac{\pi}{5} < \tan\frac{2\pi}{5}$

$\therefore \tan\frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$

Ex 4.4 $\Longrightarrow$

<u>Note:</u>