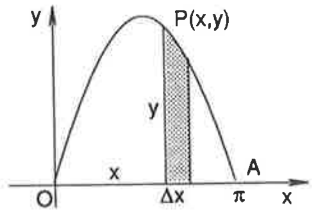
Again the integral $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ represents the area of a semi-circle of radius a, i.e. $\frac{\pi a^2}{2}$

$\therefore \quad V = 4\pi c \cdot \frac{\pi a^2}{2} = 2\pi^2 c\, a^2$ c. u.

Solution 3.3(9): The region OPAO bounded by the curve $y = \sin x$ and the x-axis, $0 \le x \le \pi$ is rotated about the y-axis.



A typical strip at P(x, y), parallel to the y-axis and of width Δx, generates a cylindrical shell of volume $\Delta V \approx 2\pi x \cdot y \cdot \Delta x$, where $y = \sin x$, x varies from 0 to π.

The required volume is $V = 2\pi \int_0^{\pi} xy\, dx$

$= 2\pi \int_0^{\pi} x \sin x\, dx$. Integrating by parts

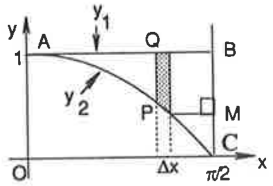
$V = 2\pi[-x \cos x + \sin x]_0^{\pi} = 2\pi \cdot (\pi)$

$\therefore \quad V = 2\pi^2$ c.u.

Solution 3.3(10): The region ACBA bounded by the curve $y = \cos x$ and the lines $x = \frac{\pi}{2}$ and $y = 1$ is rotated about the line $x = \frac{\pi}{2}$.

P(x, y)

$PM = \pi/2 - x$



A typical strip at P(x,y) on the curve $y = \cos x$, parallel to the line $x = \frac{\pi}{2}$ and of width Δx generates an element of volume (a cylindrical shell). $\Delta V \approx 2\pi(y_1 - y_2) \cdot PM \cdot \Delta x$, where

$y_1 = 1,\ y_2 = \cos x,\ PM = \frac{\pi}{2} - x.$

x varies from 0 to $\frac{\pi}{2}$.

The required volume is

$$V = 2\pi \int_0^{\pi/2} (1 - \cos x)(\frac{\pi}{2} - x)\, dx$$

We use the theorem $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

$$\therefore \quad V = 2\pi \int_0^{\pi/2} (1 - \sin x) \cdot x\, dx$$

We integrate by parts as follows:

$$\int (1 - \sin x)x\, dx = \int (x - x \sin x)\, dx$$

$$= \frac{x^2}{2} + (x \cos x - \int \cos x\, dx)$$

$$= \frac{x^2}{2} + x \cos x - \sin x$$

$$\therefore \quad V = 2\pi[\frac{x^2}{2} + x \cos x - \sin x]_0^{\pi/2}$$

$$= 2\pi(\frac{\pi^2}{8} - 1) \text{ c.u.}$$

SOLUTIONS OF PROBLEMS: CHAPTER 4

SOLUTIONS: EXERCISE 4.1

Solution 4.1(1): (a) $3\sqrt{-25} = 3\sqrt{25i^2} = 15i$

(b) $3\sqrt{\frac{-1}{4}} = 3\sqrt{\frac{i^2}{4}} = \frac{3i}{2}$

(c) $\sqrt{-12} = \sqrt{12i^2} = 2\sqrt{3}\, i$

Solution 4.1(2): (a) $(3 + i) + (-2 - 3i)$

$= (3 - 2) + (1 - 3)i = 1 - 2i$

(b) $2 - i - (5 - 2i) = 2 - i - 5 + 2i = -3 + i$

(c) $(\frac{1}{2} - \frac{i}{3}) + (\frac{1}{3} + \frac{i}{2}) = (\frac{1}{2} + \frac{1}{3}) + (\frac{i}{2} - \frac{i}{3})$

$= \frac{5}{6} + \frac{i}{6}$

Solution 4.1(3): (a) $\sqrt{-3} \times \sqrt{-12}$

$= \sqrt{3}\, i \times \sqrt{12}\, i = \sqrt{36}\, i^2 = -6$

Note that $\sqrt{-3} \times \sqrt{-12} \neq \sqrt{-3 \times (-12)}$

(b) $(2i)(3i) = 6i^2 = -6$

(c) $(1 - i)(1 + i) = 1 - i^2 = 1 - (-1) = 2$

(d) $(2 - 3i)^2 = 4 - 12i + 9i^2 = 4 - 12i - 9$

$= -5 - 12i$

Solution 4.1(4): (a) $\frac{2 - 3i}{1 + 2i} = \frac{(2 - 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)}$

$= \frac{2 - 7i + 6i^2}{1 - 4i^2} = \frac{-4 - 7i}{5} = -\frac{4}{5} - \frac{7}{5}i$

(b) $\dfrac{-5-4i}{-1+i} = \dfrac{(-5-4i)(-1-i)}{(-1+i)(-1-i)}$

$= \dfrac{5+9i+4i^2}{1-i^2} = \dfrac{1+9i}{2} = \dfrac{1}{2} + \dfrac{9i}{2}$

(c) $\dfrac{3-2i}{5i} = \dfrac{3-2i}{5i} \cdot \dfrac{i}{i} = \dfrac{3i+2}{-5} = -\dfrac{2}{5} - \dfrac{3i}{5}$

(d) $i^3 = i^2 \cdot i = -i$, hence $\dfrac{1+2i}{i^3} = \dfrac{1+2i}{-i}$

$= \dfrac{(1+2i)i}{-i \times i} = -2 + i$

Solution 4.1(5): $(x - iy)^2 = -2\sqrt{3} - 2i$

$x^2 - y^2 - 2ixy = -2\sqrt{3} - 2i$

Equating the real and imaginary parts:

$x^2 - y^2 = -2\sqrt{3}$ (1)

$2xy = 2$ (2)

We find $x^2 + y^2$ as follows:

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$= 12 + 4 = 16$

$\therefore \quad x^2 + y^2 = 4$ since $x^2 + y^2 > 0$

By adding $x^2 - y^2 = -2\sqrt{3}$ (1)

and $x^2 + y^2 = 4$ (3)

we have $2x^2 = 4 - 2\sqrt{3}$

$\therefore \quad x^2 = 2 - \sqrt{3}$

$\therefore \quad x = \pm\sqrt{2 - \sqrt{3}}$

By subtracting (1) from (3):

$2y^2 = 4 + 2\sqrt{3}$

$\therefore \quad y = \pm\sqrt{2 + \sqrt{3}}$

Considering (2), we now combine:

whence $x = \sqrt{2 - \sqrt{3}}, \quad y = \sqrt{2 + \sqrt{3}}$

or $x = -\sqrt{2 - \sqrt{3}}, \quad y = -\sqrt{2 + \sqrt{3}}$

Solution 4.1(6): $z = x + iy$

$\therefore \quad \dfrac{z+1}{z-1} = \dfrac{(x+1)+iy}{(x-1)+iy}$

$= \dfrac{(x+1)+iy}{(x-1)+iy} \cdot \dfrac{(x-1)-iy}{(x-1)-iy}$

$= \dfrac{x^2 - 1 + iy(x-1-x-1) - i^2y^2}{(x-1)^2 - i^2y^2}$

$= \dfrac{x^2+y^2-1-2iy}{x^2+y^2-2x+1}$

$\therefore \quad \dfrac{z+1}{z-1} = \dfrac{x^2+y^2-1}{x^2+y^2-2x+1} - \dfrac{2y}{x^2+y^2-2x+1} \cdot i$

Solution 4.1(7): $\alpha = 2 + i, \quad \beta = \dfrac{1}{2+i} = \dfrac{2-i}{5}$

$\therefore \quad \alpha + \beta = 2 + i + \dfrac{2-i}{5} = \dfrac{12+4i}{5}$

$\alpha\beta = (2+i) \cdot \dfrac{1}{(2+i)} = 1$

The required equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - \dfrac{1}{5}(12 + 4i)x + 1 = 0$

or $5x^2 - 4(3 + i)x + 5 = 0$

Solution 4.1(8): $\dfrac{1}{z} = 1 + i + \dfrac{2}{1-i}$

$\therefore \quad \dfrac{1}{z} = 1 + i + \dfrac{2(1+i)}{2} = 1 + i + 1 + i$

$\dfrac{1}{z} = 2 + 2i$

$z = \dfrac{1}{2+2i} = \dfrac{2-2i}{4-4i^2}$

$\therefore \quad z = \dfrac{2-2i}{8} = \dfrac{1}{4} - \dfrac{i}{4}$

Solution 4.1(9): $(2+i)z + (2-i)w = 1$ (1)

$(2-i)z + (2+i)w = 2$ (2)

Multiply (1) by $2+i$ and (2) by $2-i$ and then subtract:

$(2+i)^2z - (2-i)^2z = 2 + i - 2(2-i)$

$z(4 + 4i + i^2 - 4 + 4i - i^2) = -2 + 3i$

$8iz = -2 + 3i$

Multiply both sides by i

$\therefore \quad -8z = -2i - 3$

$\therefore \quad z = \dfrac{3}{8} + \dfrac{1}{4}i$

Multiply (1) by $(2-i)$, (2) by $(2+i)$ and subtract, giving

$w[(2-i)^2 - (2+i)^2] = 2 - i - 2(2+i)$

$\therefore \quad -8wi = -2 - 3i$

Multiply by i, then $8w = -2i + 3$

$w = \dfrac{3}{8} - \dfrac{i}{4}$

Solution 4.1(10): $z = 1 + i$

(a) $z^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$

(b) $z^4 = z^2 \cdot z^2 = (2i) \cdot (2i) = -4$

(c) $z^2 + \dfrac{1}{z^2} = 2i + \dfrac{1}{2i} = 2i + \dfrac{1}{2} \cdot \dfrac{i}{i^2}$

$= 2i - \dfrac{1}{2}i = \dfrac{3}{2}i$

SOLUTIONS: EXERCISE 4.2

Solution 4.2(1):

(a) $z = 4 = 4 \times 1 = 4(\cos 0 + i \sin 0)$

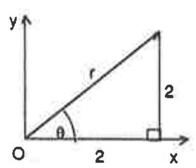
(b) $z = -4 = 4 \times (-1) = 4(\cos \pi + i \sin \pi)$

(c) $z = 4i = 4(\cos \dfrac{\pi}{2} + i \sin \dfrac{\pi}{2})$

(d) $z = 2 + 2i$

$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$\tan \theta = \dfrac{2}{2} = 1$

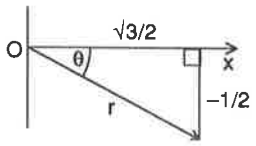


θ is in the first quadrant $\theta = 45° = \dfrac{\pi}{4}$

$\therefore \quad z = r \text{ cis } \theta = 2\sqrt{2} \text{ cis } (\dfrac{\pi}{4})$

(e)

$z = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$



$|z| = \sqrt{(\dfrac{\sqrt{3}}{2})^2 + (-\dfrac{1}{2})^2} = 1$

$\tan \theta = \dfrac{-1/2}{\sqrt{3}/2} = -\dfrac{1}{\sqrt{3}}$

θ is in the fourth quadrant.

$\therefore \quad \theta = -30° = -\dfrac{\pi}{6}$

$z = r \text{ cis } \theta = 1 \text{ cis } (-\dfrac{\pi}{6})$

Solution ·4.2(2):(a) $z = 2(\cos \dfrac{\pi}{3} + i \sin \dfrac{\pi}{3})$

$= 2(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i) = 1 + \sqrt{3}i$

(b) $z = 2[\cos(-\dfrac{2\pi}{3}) + i \sin(-\dfrac{2\pi}{3})]$

$= 2(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i) = -1 - \sqrt{3}i$

Solution 4.2(3):(a) Let $z = [2 \text{ cis } (\dfrac{3\pi}{4})]^2$

By De Moivre's theorem,

$z = 4 \text{ cis } (2 \times \dfrac{3\pi}{4}) = 4 \text{ cis } (\dfrac{3\pi}{2})$ (1)

Now $\cos(\dfrac{\pi}{4}) - i \sin(\dfrac{\pi}{4})$

$= \cos(-\dfrac{\pi}{4}) + i \sin(-\dfrac{\pi}{4}) = \text{cis } (-\dfrac{\pi}{4})$

Let $w = [5 \text{ cis } (-\dfrac{\pi}{4})]^2$

By De Moivre's theorem,

$w^2 = 25 \text{ cis } (-2 \times \dfrac{\pi}{4}) = 25 \text{ cis } (-\dfrac{\pi}{2})$

The given expression is:

$zw = 4 \text{ cis } (\dfrac{3\pi}{2}) \cdot 25 \text{ cis } (-\dfrac{\pi}{2})$

$= 100 \text{ cis } (\dfrac{3\pi}{2} + \dfrac{-\pi}{2}) = 100 \text{ cis } \pi = -100$

(b) Using the results of part (a), the given expression is

$\dfrac{z}{w} = 4 \text{ cis } (\dfrac{3\pi}{2}) \div [25 \text{ cis}(-\dfrac{\pi}{2})]$

$= \dfrac{4}{25} \text{ cis } (\dfrac{3\pi}{2} - \dfrac{-\pi}{2}) = \dfrac{4}{25} \text{ cis } (2\pi)$

$= \dfrac{4}{25}$

Solution 4.2(4): (a) $\sqrt{3} + i = 2 \text{ cis } (\dfrac{\pi}{6})$

(b) $1 - i = \sqrt{2} \text{ cis } (-\dfrac{\pi}{4})$

(c) $1 + \sqrt{3}i = 2 \text{ cis } (\dfrac{\pi}{3})$

(d) $\sqrt{3} - i = 2 \text{ cis } (-\dfrac{\pi}{6})$

Then $\dfrac{(\sqrt{3}+i)(1-i)}{(1+\sqrt{3}i)(\sqrt{3}-i)}$

$= \dfrac{2 \text{ cis } (\pi/6) \cdot \sqrt{2} \text{ cis } (-\pi/4)}{2 \text{ cis } (\pi/3) \cdot 2 \text{ cis } (-\pi/6)}$

$= \dfrac{1}{\sqrt{2}} \text{cis } (\dfrac{\pi}{6} + \dfrac{-\pi}{4}) \div \text{cis}(\dfrac{\pi}{3} + \dfrac{-\pi}{6})$

$= \dfrac{1}{\sqrt{2}} \text{cis } [\dfrac{\pi}{6} - \dfrac{\pi}{4} - (\dfrac{\pi}{3} - \dfrac{\pi}{6})] = \dfrac{1}{\sqrt{2}} \text{cis } (-\dfrac{\pi}{4})$

Solution 4.2(5): We have, by using

$z_1z_2 = r_1r_2 \text{ cis } (\theta_1 + \theta_2)$:

$r \text{ cis } \theta \cdot r \text{ cis } \theta = r^2 \text{ cis } (2\theta)$ (1)

Then $(r \text{ cis } \theta)^3 = r \text{ cis } \theta \cdot r \text{ cis } \theta \cdot r \text{ cis } \theta$

Using (1), $(r \text{ cis } \theta)^3 = r \text{ cis } \theta \cdot r^2 \text{ cis } 2\theta$

$= r^3 \text{ cis } (\theta + 2\theta) = r^3 \text{ cis } 3\theta$

SOLUTIONS: EXERCISE 4.3

Solution 4.3(1): $z = 1 + i = \sqrt{2} \text{ cis } (\dfrac{\pi}{4})$

By De Moivre's theorem,

$z^8 = (\sqrt{2})^8 \text{ cis } (\dfrac{8\pi}{4}) = 16 \text{ cis } 2\pi$

$= 16$

By De Moivre's theorem for the negative index

$z^{-8} = (\sqrt{2})^{-8} \text{ cis } (-\dfrac{8\pi}{4}) = \dfrac{1}{16} \text{ cis } (-2\pi) = \dfrac{1}{16}$

or $z^{-8} = \dfrac{1}{z^8} = \dfrac{1}{16}$

Solution 4.3(2): By De Moivre's theorem, the numerator is equal to

$3^6 [\cos(6 \times \dfrac{\pi}{6}) + i \sin(6 \times \dfrac{\pi}{6})]$

$= 729 (\cos \pi + i \sin \pi) = 729 \text{ cis } \pi$

Denominator $= 2^2 \text{ cis } (2 \times \dfrac{\pi}{3}) = 4 \text{ cis } (\dfrac{2\pi}{3})$

Hence the given expression is

$\dfrac{729 \text{ cis } \pi}{4 \text{ cis } (2\pi/3)} = \dfrac{729}{4} \text{ cis } (\pi - \dfrac{2\pi}{3})$

$= \dfrac{729}{4} \text{ cis } (\dfrac{\pi}{3})$

Solution 4.3(3): Let $z = 1 + \sqrt{3}i = 2 \text{ cis } (\dfrac{\pi}{3})$

and $w = 1 - i = \sqrt{2} \text{ cis } (-\dfrac{\pi}{4})$

Continued

By De Moivre's theorem:

$$z^3 = 2^3 \text{ cis } \left(\frac{3\pi}{3}\right) = 8 \text{ cis } \pi$$

and $w^4 = (\sqrt{2})^4 \text{ cis } \left(-\frac{4\pi}{4}\right) = 4 \text{ cis } (-\pi)$

Hence the given expression

$$= \frac{z^3}{w^4} = 8 \text{ cis } \left(\frac{\pi}{2}\right) \div 4 \text{ cis } (-\pi)$$

$$= 2 \text{ cis } [\pi - (-\pi)] = 2 \text{ cis } (2\pi) = 2$$

Solution 4.3(4): Using $c = \cos\theta$, $s = \sin\theta$,

we have: $\cos 5\theta + i \sin 5\theta$

$$= (\cos\theta + i\sin\theta)^5 \quad \text{(De Moivre's th.)}$$
$$= (c + is)^5$$
$$= c^5 + 5c^4is + 10c^3i^2s^2 + 10c^2i^3s^3 + 5ci^4s^4 + i^5s^5$$
$$= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$$
$$= c^5 - 10c^3s^2 + 5cs^4 + i(5c^4s - 10c^2s^3 + s^5)$$

Equating the real and the imaginary parts:

$$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4 \quad (1)$$
$$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5 \quad (2)$$

where $c = \cos\theta$ and $s = \sin\theta$.

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$$

Remembering that $\frac{\sin\theta}{\cos\theta} = \tan\theta$,

we divide both the numerator and denominator by c^5, i.e. $\cos^5\theta$, then:

$$\tan 5\theta = \frac{5\frac{c^4s}{c^5} - 10\frac{c^2s^3}{c^5} + \frac{s^5}{c^5}}{\frac{c^5}{c^5} - 10\frac{c^3s^2}{c^5} + 5\frac{cs^4}{c^5}}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

Solution 4.3(5): Let $z = \text{cis }\theta$,

so $\frac{1}{z} = \text{cis }(-\theta)$, $z^n = \text{cis } n\theta$, and $z^{-n} = \text{cis }(-n\theta)$

Now $z^{-n} = \frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta)$

$$= \cos n\theta - i \sin n\theta$$

$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$\therefore \quad z^n + \frac{1}{z^n} = 2\cos n\theta \quad (1)$$

For $n = 1$, $z + \frac{1}{z} = 2\cos\theta$ (2)

$$\therefore \quad (2\cos\theta)^6 = \left(z + \frac{1}{z}\right)^6$$

We expand $\left(z + \frac{1}{z}\right)^6$ by the Binomial theorem:

$$\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5 \cdot \frac{1}{z} + 15z^4 \cdot \frac{1}{z^2} + 20z^3 \cdot \frac{1}{z^3} + 15z^2 \cdot \frac{1}{z^4} + 6z \cdot \frac{1}{z^5} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

Put $n = 6, 4, 2$ in (1), then:

$$2^6\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

Divide by 64

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

$$\therefore \quad \int_0^{\pi/2} \cos^6\theta \, d\theta$$

$$= \frac{1}{32}\int_0^{\pi/2} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$$

$$= \frac{1}{32}\left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\pi/2}$$

$$= \frac{1}{32}(5\pi) = \frac{5\pi}{32}$$

Solution 4.3(6): Using $1 + \cos 2\theta = 2\cos^2\theta$

and $\sin 2\theta = 2\sin\theta\cos\theta$,

$$(1 + \cos 2\theta + i\sin 2\theta)^n$$
$$= (2\cos^2\theta + 2i\sin\theta\cos\theta)^n$$
$$= [2\cos\theta(\cos\theta + i\sin\theta)]^n$$
$$= 2^n\cos^n\theta(\cos\theta + i\sin\theta)^n$$
$$= 2^n\cos^n\theta(\cos n\theta + i\sin n\theta)$$

where we used De Moivre's theorem

$$(\text{cis }\theta)^n = \text{cis } n\theta$$

Solution 4.3(7): $(1 - z^4) = (1 - z^2)(1 + z^2)$

$$= (1 - z)(1 + z)(1 + z^2)$$
$$= (1 - z)(1 + z + z^2 + z^3)$$

Hence dividing by $1 - z$, we have

$$1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z} \quad (1)$$

$z = \text{cis }\theta$

$\therefore$ by De Moivre's theorem

L.H.S. $= 1 + \text{cis }\theta + \text{cis } 2\theta + \text{cis } 3\theta$

The real part of the L.H.S. is

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta \quad (2)$$

Now the R.H.S.:

$$\frac{1 - z^4}{1 - z} = \frac{1 - (\cos 4\theta + i\sin 4\theta)}{1 - (\cos\theta + i\sin\theta)}$$

We use: $1 - \cos 2A = 2\sin^2 A$ and

$\sin 2A = 2\sin A\cos A$

Then

$$1 - \cos 4\theta - i\sin 4\theta$$
$$= 2\sin^2 2\theta - 2i\sin 2\theta\cos 2\theta$$
$$= 2\sin 2\theta(\sin 2\theta - i\cos 2\theta)$$
$$= 2\sin 2\theta(\cos(\pi/2 - 2\theta) - i\sin(\pi/2 - 2\theta))$$
$$= 2\sin 2\theta \text{ cis }(2\theta - \pi/2)$$

Similarly, $1 - \cos\theta - i\sin\theta$

$$= 2\sin\left(\frac{\theta}{2}\right)\text{cis}\left(\frac{\theta}{2} - \frac{\pi}{2}\right)$$

Continued

Hence the R.H.S. is

$$\frac{2\sin 2\theta}{2\sin(\theta/2)} \cdot \frac{\text{cis }(2\theta - \pi/2)}{\text{cis }(\theta/2 - \pi/2)}$$

$$= \frac{2\sin 2\theta}{2\sin(\theta/2)} \cdot \text{cis}\left[2\theta - \frac{\pi}{2} - \left(\frac{\theta}{2} - \frac{\pi}{2}\right)\right]$$

$$= \frac{2\sin 2\theta}{2\sin(\theta/2)} \cdot \text{cis}\left(\frac{3\theta}{2}\right) \quad (3)$$

The real part of (3) is

$$\frac{2\sin 2\theta\cos(3\theta/2)}{2\sin(\theta/2)} \qquad \boxed{\text{Using } 2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{\sin(2\theta + 3\theta/2) + \sin(2\theta - 3\theta/2)}{2\sin(\theta/2)}$$

$$= \left[\sin\left(\frac{7\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right] \div 2\sin\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left[1 + \frac{\sin(7\theta/2)}{\sin(\theta/2)}\right] \quad (4)$$

(2) and (4) give the required result, i.e.

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta$$
$$= \frac{1}{2}\left[1 + \frac{\sin(7\theta/2)}{\sin(\theta/2)}\right]$$

Solution 4.3(8): $\sqrt{3} + i = 2\text{ cis}\left(\frac{\pi}{6}\right)$ and

$1 + i = \sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$

$$\therefore \quad z = \frac{\sqrt{3} + i}{1 + i} = \frac{2\text{ cis}\left(\frac{\pi}{6}\right)}{\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)}$$

$$\therefore \quad z = \sqrt{2}\text{ cis}\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = \sqrt{2}\text{ cis}\left(-\frac{\pi}{12}\right)$$

By De Moivre's theorem:

$$z^n = (\sqrt{2})^n\text{ cis}\left(-\frac{n\pi}{12}\right) \quad (1)$$

$$= (\sqrt{2})^n\left[\cos\left(-\frac{n\pi}{12}\right) + i\sin\left(-\frac{n\pi}{12}\right)\right]$$

If z^n is real, then $\sin\left(-\frac{n\pi}{12}\right) = 0$

The smallest positive value of n is given by

$$-\frac{n\pi}{12} = -\pi, \text{ i.e. } n = 12$$

Then from (1), substituting $n = 12$,

$$z^{12} = (\sqrt{2})^{12}\cos(-\pi) = -64$$

Solution 4.3(9): Let $c = \cos\theta$ and $s = \sin\theta$

Now consider the expression:

$$(1 + s - ic)(s + ic) = s + ic + (s^2 - i^2c^2)$$
$$= s + ic + s^2 + c^2, \qquad (s^2 + c^2 = 1)$$
$$= s + ic + 1$$

Hence, $1 + s + ic = (1 + s - ic)(s + ic)$

$$\therefore \quad \frac{1 + s + ic}{1 + s - ic} = s + ic$$

Thus,

$$\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta \quad (1)$$

Put $\theta = \frac{\pi}{5}$ and raise both sides to power of 5,

then: $\frac{(1 + \sin\pi/5 + i\cos\pi/5)^5}{(1 + \sin\pi/5 - i\cos\pi/5)^5}$

$$= (\sin\pi/5 + i\cos\pi/5)^5$$

R.H.S. of the last relation is

$$= \left[\text{cis}\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\right]^5 = \left(\text{cis }\frac{3\pi}{10}\right)^5 = \text{cis}\left(\frac{3\pi}{2}\right)$$

$$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$$

Hence, upon cross-multiplication and transposition the result follows.

Solution 4.3(10): $\sqrt{3} + i = 2\text{ cis}\left(\frac{\pi}{6}\right)$ and

$\sqrt{3} - i = 2\text{ cis}\left(-\frac{\pi}{6}\right)$

Then $(\sqrt{3} + i)^m = (\sqrt{3} - i)^m$ becomes:

$$2^m\left(\text{cis }\frac{\pi}{6}\right)^m = 2^m\left(\text{cis } -\frac{\pi}{6}\right)^m$$

$$\therefore \quad \left(\text{cis }\frac{\pi}{6}\right)^m = \left(\text{cis } -\frac{\pi}{6}\right)^m$$

By De Moivre's theorem:

$$\cos\left(\frac{m\pi}{6}\right) + i\sin\left(\frac{m\pi}{6}\right)$$
$$= \cos\left(-\frac{m\pi}{6}\right) + i\sin\left(-\frac{m\pi}{6}\right)$$

The R.H.S. $= \cos\left(\frac{m\pi}{6}\right) - i\sin\left(\frac{m\pi}{6}\right)$

$$\therefore \quad 2i\sin\left(\frac{m\pi}{6}\right) = 0$$

$$\therefore \quad \sin\left(\frac{m\pi}{6}\right) = 0$$

The smallest positive integer m is given by

$$\frac{m\pi}{6} = \pi$$

$$\therefore \quad m = 6$$

SOLUTIONS: EXERCISE 4.4

Solution 4.4(1):(a) Let $z^2 = i$, where $z = x + iy$

$$\therefore \quad x^2 - y^2 + 2ixy = i$$

Equating the real and imaginary parts:

$$x^2 - y^2 = 0 \quad (1)$$
$$2xy = 1 \quad (2)$$
$$\therefore \quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 1$$
$$x^2 + y^2 = 1 \quad (3)$$

Adding (1) and (3):

$$2x^2 = 1 \text{ gives } x = \pm\frac{1}{\sqrt{2}}$$

Continued

From (2): when $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$,

when $= -\frac{1}{\sqrt{2}}$, $y = -\frac{1}{\sqrt{2}}$

Hence the required roots are: $\pm(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}})$

(b) Let $z^2 = 2 + 2i$, where $z = x + iy$

$\therefore \quad x^2 - y^2 + 2ixy = 2 + 2i$

Equating the real and imaginary parts:

$$x^2 - y^2 = 2 \qquad (1)$$
$$2xy = 2 \qquad (2)$$

Then $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 4 + 4$

$$x^2 + y^2 = 2\sqrt{2}, \quad x^2 + y^2 > 0 \qquad (3)$$

Add (1) and (3):

$$2x^2 = 2 + 2\sqrt{2} \text{ gives } x^2 = 1 + \sqrt{2}$$

Subtracting (1) and (2):

$$2y^2 = 2\sqrt{2} - 2 \text{ gives } y^2 = \sqrt{2} - 1$$

From (2) we find that when $x > 0$, y is > 0 and when $x < 0$, $y < 0$.

$\therefore$ The required square roots are

$$\pm[\sqrt{\sqrt{2} + 1} + i\sqrt{\sqrt{2} - 1}]$$

Solutions 4.4(2): $\sqrt{3} + i = 2 \text{ cis } (\frac{\pi}{6})$

$$= 2 \text{ cis } (2k\pi + \frac{\pi}{6})$$

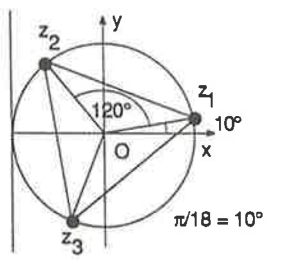
The cube roots are given by

$$z = \sqrt[3]{2} \text{ cis } (\frac{2k\pi}{3} + \frac{\pi}{18}), \ k = 0, 1, 2$$

The required cube roots are

$$z_1 = \sqrt[3]{2} \text{ cis } (\frac{\pi}{18}), \ z_2 = \sqrt[3]{2} \text{ cis } (\frac{13\pi}{18}),$$

and $z_3 = \sqrt[3]{2} \text{ cis } (\frac{25\pi}{18})$



These roots lie on a circle of radius $\sqrt[3]{2}$, centre at O(0,0), the angular separation between any two roots being 120 °.

The roots z_1, z_2, z_3 form an equilateral triangle whose area is $3 \times \frac{1}{2} \times \sqrt[3]{2} \cdot \sqrt[3]{2} \sin 120°$

$$= \frac{3\sqrt{3}}{4} \cdot 2^{2/3} = \frac{3\sqrt{3}}{2^{4/3}}$$

Solution 4.4(3): $z^2 - (2 - i)z - 2i = 0$

$a = 1, \quad b = -2 + i, \quad c = -2i$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 - i \pm \sqrt{(-2+i)^2 + 8i}}{2}$$
$$= \frac{2 - i \pm \sqrt{3 + 4i}}{2}$$

Now $3 + 4i = 4 + 2\times2i + i^2 = (2 + i)^2$

Hence $z = \frac{2 - i \pm (2 + i)}{2}$

The roots are $z = 2, -i$

Solution 4.4(4): $z^6 = -i = \text{cis } (-\frac{\pi}{2})$

$$= \text{cis } (2k\pi - \frac{\pi}{2})$$

Hence the six roots are given by:

$$z = \text{cis } (\frac{k\pi}{3} - \frac{\pi}{12}), \ k = 0, 1, 2, 3, 4, 5$$

$$z_1 = \text{cis } (-\frac{\pi}{12}) = \text{cis } (-15°) \quad \boxed{\frac{360°}{6} = 60°}$$

$$z_2 = \text{cis } (\frac{\pi}{4}) = \text{cis } 45°$$

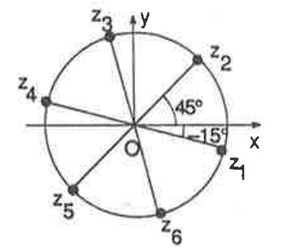
$$z_3 = \text{cis } (\frac{7\pi}{12}) = \text{cis } 105°$$

$$z_4 = \text{cis } (\frac{11\pi}{12}) = \text{cis } 165°$$

$$z_5 = \text{cis } (\frac{15\pi}{12}) = \text{cis } 225°$$

$$z_6 = \text{cis } (\frac{19\pi}{12}) = \text{cis } 285°$$

The roots lie on a circle of radius 1. They are equally spaced, with an angular separation of $360° \div 6 = 60°$.



SOLUTIONS: EXERCISE 4.5

Solution 4.5(1):

(a) Let $z = x + iy$,

then $\quad \bar{z} = x - iy$

$\therefore \quad \bar{z} + ai = x - iy + ai = x + (a-y)i$

$\therefore \quad \overline{\bar{z} + ai} = x - (a - y)i = x + iy - ai$
$= z - ai$

(b) $\frac{\overline{(2 + i)^2}}{3 - 4i} = \frac{(2 - i)^2}{3 - 4i} = \frac{4 - 4i + i^2}{3 - 4i}$

$$= \frac{3 - 4i}{3 - 4i} = 1$$

(c) Let $w = z_2z_3$

We use the following two properties of the conjugate complex numbers, namely:

$$\overline{w_1w_2} = \bar{w}_1 \cdot \bar{w}_2 \qquad \text{and}$$

$$\overline{(\frac{w_1}{w_2})} = \frac{\bar{w}_1}{\bar{w}_2}$$

We have: $\overline{(\frac{z_1}{z_2z_3})} = \overline{(\frac{z_1}{w})} = \frac{\bar{z}_1}{\bar{w}} = \frac{\bar{z}_1}{\overline{z_1z_2}}$

$$= \frac{\bar{z}_1}{\bar{z}_2 \cdot \bar{z}_3}$$

Solution 4.5(2): $z = x + iy$ and $\omega = 1 + i$

Using $z\bar{z} = x^2 + y^2$,

$$\omega\bar{\omega} = (1 + i)(1 - i) = 1 - i^2 = 2$$

$\therefore \quad (z - \omega)(\bar{z} - \bar{\omega}) = 1$ gives

$$z\bar{z} - z\bar{\omega} - \bar{z}\omega + \omega\bar{\omega} = 1$$

$\therefore \quad x^2 + y^2 - (x + iy)(1 - i)$
$\qquad - (x - iy)(1 + i) + 2 = 1$

This simplifies to:

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

or $\quad (x - 1)^2 + (y - 1)^2 = 1 \qquad (1)$

Hence the equation $(z - \omega)(\bar{z} - \bar{\omega}) = 1$ represents a circle of radius 1 and the centre at (1, 1).

Alternatively

$(z - \omega)(\bar{z} - \bar{\omega}) = 1$ is the same as

$$|z - \omega|^2 = 1$$

$\therefore \quad |x + iy - 1 - i| = 1$

$$\sqrt{(x - 1)^2 + (y - 1)^2} = 1$$

This is the same as relation (1).

Solution 4.5(3): $z = \text{cis } \theta$

$\therefore \quad \frac{1}{z} = \frac{1}{\text{cis } \theta} = \frac{\text{cis } (-\theta)}{\text{cis } \theta \cdot \text{cis } (-\theta)}$

$$\frac{1}{z} = \frac{\text{cis } (-\theta)}{\text{cis } 0} = \frac{\text{cis } (-\theta)}{1} = \text{cis } (-\theta)$$

$\therefore \quad \frac{1}{z} = \cos(-\theta) + i \sin(-\theta) = \cos\theta - i \sin\theta$

$$(1)$$

So $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \cos\alpha - i \sin\alpha$
$\qquad + \cos\beta - i \sin\beta + \cos\gamma - i \sin\gamma$

$$= (\cos\alpha + \cos\beta + \cos\gamma)$$
$$-i(\sin\alpha + \sin\beta + \sin\gamma) \qquad (2)$$

Now we are given that:

$$z_1 + z_2 + z_3 = 0$$

$\therefore \quad \text{cis } \alpha + \text{cis } \beta + \text{cis } \gamma = 0$

Equate the real and the imaginary parts:

$$\cos\alpha + \cos\beta + \cos\gamma = 0$$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$

Hence (2) gives $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0 - i \cdot 0 = 0$

Solution 4.5(4):

(a) We use $1 - \cos\theta = 2 \sin^2(\frac{\theta}{2})$

and $\sin\theta = 2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})$

$\therefore \quad 1 - \cos\theta + 2i \sin\theta$

$$= 2\sin(\frac{\theta}{2})[\sin(\frac{\theta}{2}) + 2i \cos(\frac{\theta}{2})]$$

(b) $(1 - \cos\theta + 2i \sin\theta)^{-1}$, using (a)

$$= \frac{1}{2 \sin(\frac{\theta}{2})[\sin(\frac{\theta}{2}) + 2i \cos(\frac{\theta}{2})]}$$

$$= \frac{\sin(\frac{\theta}{2}) - 2i \cos(\frac{\theta}{2})}{2 \sin(\frac{\theta}{2})[\sin^2(\frac{\theta}{2}) + 4 \cos^2(\frac{\theta}{2})]}$$

Now $\sin^2(\frac{\theta}{2}) + 4 \cos^2(\frac{\theta}{2})]$

$$= \frac{1 - \cos\theta}{2} + 2(1 + \cos\theta)]$$

$$= \frac{5 + 3\cos\theta}{2}$$

$\therefore \quad (1 - \cos\theta + 2i \sin\theta)^{-1}$

$$= \frac{\sin(\frac{\theta}{2}) - 2i \cos(\frac{\theta}{2})}{\sin(\frac{\theta}{2})(5 + 3\cos\theta)}$$

$$= \frac{1 - 2i \cot(\frac{\theta}{2})}{5 + 3\cos\theta}$$

Solution 4.5(5): $z^2 = i\bar{z}$

Substituting $z = x + iy$, $\bar{z} = x - iy$

$$x^2 - y^2 + 2ixy = i(x - iy) = y + xi$$

Equating the real and the imaginary parts:

$$x^2 - y^2 = y \qquad (1)$$
$$2xy = x \qquad \text{(Do not cancel x)} \qquad (2)$$

From (2) $\qquad x(2y - 1) = 0$

$\therefore \quad x = 0$ or $y = \frac{1}{2}$

When $x = 0$ from (1): $-y^2 = y$,

Continued

i.e. $y = 0$ or $y = -1$

So, $(x, y) = (0, 0)$ or $(0, -1)$ (3)

Again when $y = \frac{1}{2}$ from (1):

$$x^2 - \frac{1}{4} = \frac{1}{2} \quad \text{gives } x = \pm\frac{\sqrt{3}}{2}$$

So, $(x, y) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ or $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

(4)

Thus the complete solution of $z^2 = i\bar{z}$ is:

$$(x, y) = (0, 0), (0, -1), (\pm\frac{\sqrt{3}}{2}, \frac{1}{2})$$

SOLUTIONS: EXERCISE 4.6

Solution 4.6(1): $z^6 = 1 = \text{cis}\,(2k\pi)$

The six roots are given by

$$z = \text{cis}\,(\frac{2k\pi}{6}),\ k = 0, 1, 2, 3, 4, 5$$

These are:

$$z_1 = \text{cis}\,0,\ z_2 = \text{cis}\,(\frac{\pi}{3}),\ z_3 = \text{cis}\,(\frac{2\pi}{3})$$

$$z_4 = \text{cis}\,\pi,\ z_5 = \text{cis}\,(\frac{4\pi}{3}),\ z_6 = \text{cis}\,(\frac{5\pi}{3})$$

Since ω is the complex root of $z^6 = 1$ with the smallest argument, $\omega = z_2 = \text{cis}\,(\frac{\pi}{3})$, then by De Moivre's theorem:

$$\omega^2 = \text{cis}\,(\frac{2\pi}{3}) = z_3$$

$$\omega^3 = \text{cis}\,\pi = -1 = z_4 \text{ (real)}$$

$$\omega^4 = \text{cis}\,(\frac{4\pi}{3}) = z_5$$

$$\omega^5 = \text{cis}\,(\frac{5\pi}{3}) = z_6$$

(a) We have

$z^6 - 1 = (z - 1)(z + 1)(z^4 + z^2 + 1) = 0$

Since $z = \pm 1$ are the real roots, the remaining four complex roots ω, ω^2, ω^4 and ω^5 must be the roots of $z^4 + z^2 + 1 = 0$

(b) $\alpha = \omega + \omega^5 = \text{cis}\,(\frac{\pi}{3}) + \text{cis}\,(\frac{5\pi}{3})$

$$= \text{cis}\,(\frac{\pi}{3}) + \text{cis}\,(-\frac{\pi}{3}) = 2\cos\,(\frac{\pi}{3}) = 1$$

$$\beta = \omega^2 + \omega^4 = \text{cis}\,(\frac{2\pi}{3}) + \text{cis}\,(\frac{4\pi}{3})$$

$$\therefore \beta = \text{cis}\,(\frac{2\pi}{3}) + \text{cis}\,(-\frac{2\pi}{3}) = 2\cos\,(\frac{2\pi}{3}) = -1$$

Hence the required equation is

$(x - 1)(x + 1) = 0$, i.e., $x^2 - 1 = 0$

Solution 4.6(2): $z = a + ib$ is a solution of $z^3 - 1 = 0$, so it satisfies this equation.

$\therefore$ $(a + ib)^3 = 1$

$$a^3 + i^3b^3 + 3iab(a + ib) = 1$$

$$(a^3 - 3ab^2) + (3a^2b - b^3)\,i = 1 + 0{\cdot}i$$

Equating the real and the imaginary parts:

$a^3 - 3ab^2 = 1$ (1)

$3a^2b - b^3 = 0$ (2)

Let $f(z) = z^3 - 1$, then $f(a - ib) = (a - ib)^3 - 1$

$= a^3 - i^3b^3 - 3iab(a - ib) - 1$

$= a^3 - 3ab^2 - (3a^2b - b^3)\,i - 1$

Using (1) and (2)

$f(a - ib) = 1 - 0{\cdot}i - 1 = 0$

Hence by the remainder theorem, $a - ib$ is a root of $z^3 - 1 = 0$.

We are not asked to solve $z^3 - 1 = 0$, but if we are asked, from (2): $b = 0$ or $3a^2 = b^2$ (3)

From (1), $a^3 = 1$, i.e. $a = 1$

So one root is $z = a + ib = 1$

If $b^2 = 3a^2$, using (1), $a^3 - 9a^3 = 1$ or $a^3 = -\frac{1}{8}$

gives $a = -\frac{1}{2}$

Hence $b^2 = 3a^2$ gives $b = \pm\frac{\sqrt{3}}{2}$.

So the three roots are $z = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i$.

It is an interesting and instructive method.

Solution 4.6(3): (a) We have

$z^5 = 1 = \text{cis}\ 2k\pi$ (1)

Let $z = r\,\text{cis}\,\theta$ be a root of $z^5 - 1 = 0$, then by De Moivre's theorem: $z^5 = r^5\text{cis}\,5\theta$ (2)

From (1) and (2): $r^5\,\text{cis}\,5\theta = \text{cis}\,2k\pi$

Equating moduli: $r^5 = 1$, i.e. $r = 1$

Equating arguments: $5\theta = 2k\pi$ or $\theta = \frac{2k\pi}{5}$

The five roots are given by $k = 0, 1, 2, 3, 4$

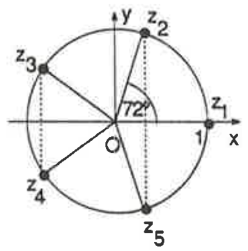
$$z_1 = \text{cis}\,0 = 1,\ z_2 = \text{cis}\,(\frac{2\pi}{5}) = \text{cis}\,72^\circ$$

$$z_3 = \text{cis}\,(\frac{4\pi}{5}) = \text{cis}\,144^\circ,$$

$$z_4 = \text{cis}\,(\frac{6\pi}{5}) = \text{cis}\,(-\frac{4\pi}{5}) = \text{cis}\,216^\circ$$

$$z_5 = \text{cis}\,(\frac{8\pi}{5}) = \text{cis}\,(-\frac{2\pi}{5}) = \text{cis}\,288^\circ$$

The roots lie on a circle of radius 1, centre O(0, 0) with angular separation of 72° between any two consecutive roots.



(b) $(z - 1)(z^4 + z^3 + z^2 + z + 1)$

$= z(z^4 + z^3 + z^2 + z + 1)$

$\quad - 1(z^4 + z^3 + z^2 + z + 1)$

$= z^5 + z^4 + z^3 + z^2 + z$

$\quad - z^4 - z^3 - z^2 - z - 1$

$= z^5 - 1$

(c) To solve $z^4 + z^3 + z^2 + z + 1 = 0$ we use the result of (a) and (b). Since $z^5 - 1$

$= (z - 1)(z^4 + z^3 + z^2 + z + 1)$

the five roots of $z^5 - 1 = 0$ are given by $z = 1$ and the four roots of $z^4 + z^3 + z^2 + z + 1 = 0$.

In part (a), we obtained all five roots.

We deduce that the roots of

$z^4 + z^3 + z^2 + z + 1 = 0$ are:

$$\text{cis}\,(\frac{2\pi}{5}),\ \text{cis}\,(\frac{4\pi}{5}),\ \text{cis}\,(-\frac{2\pi}{5}) \text{ and } \text{cis}\,(-\frac{4\pi}{5})$$

(d) $z^4 + z^3 + z^2 + z + 1 = 0$

Divide by z^2, then

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$$

or $$(z^2 + \frac{1}{z^2}) + (z + \frac{1}{z}) + 1 = 0 \quad (1)$$

Let $z = \text{cis}\,\theta$, then by De Moivre's theorem for positive and negative indices:

$$z + \frac{1}{z} = z + z^{-1} = \text{cis}\,\theta + \text{cis}\,(-\theta) = 2\cos\theta$$

$$z^2 + \frac{1}{z^2} = z^2 + z^{-2} = \text{cis}\,2\theta + \text{cis}\,(-2\theta)$$

$= 2\cos 2\theta$

Hence (1) becomes:

$2\cos 2\theta + 2\cos\theta + 1 = 0$

$2(2\cos^2\theta - 1) + 2\cos\theta + 1 = 0$

$4\cos^2\theta + 2\cos\theta - 1 = 0$

We write $c = \cos\theta$, then:

$4c^2 + 2c - 1 = 0$ (2)

The two roots of (2) are given by $\cos\theta$, where

$$\theta = \frac{2k\pi}{5},\quad k = 1, 2, 3, 4.$$

Now $\cos\,(\frac{2\pi}{5}) = \cos\,(\frac{8\pi}{5})$

and $\cos\,(\frac{4\pi}{5}) = \cos\,(\frac{6\pi}{5})$.

Thus (2) has two distinct roots, $\cos\,(\frac{2\pi}{5})$ and $\cos\,(\frac{4\pi}{5})$.

(e) The sum of the roots of (2) is

$$\cos\,(\frac{2\pi}{5}) + \cos\,(\frac{4\pi}{5}) = -\frac{2}{4} = -\frac{1}{2}$$

SOLUTIONS: EXERCISE 4.7

Solution 4.7(1):(a) $z^3 + 1 = (z + 1)(z^2 - z + 1)$

Now $z^2 - z + 1 = (z - \frac{1}{2})^2 + \frac{3}{4}$

$$= (z - \frac{1}{2})^2 - (\frac{\sqrt{3}}{2}i)^2$$

$$= (z - \frac{1}{2} + \frac{\sqrt{3}}{2}i)(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i)$$

$$\therefore\ z^3 + 1 = (z + 1)\,(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i)$$

$$\times\,(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i)$$

(b) To factorise $z^5 + 1$ we first solve $z^5 + 1 = 0$.

$\therefore$ $z^5 = -1 = \text{cis}\,(\pi + 2k\pi)$.

The five roots are given by:

$$z = \text{cis}\,(\frac{\pi + 2k\pi}{5}),\ k = 0, 1, 2, 3, 4$$

These are: $z_1 = \text{cis}\,(\frac{\pi}{5})$, $z_2 = \text{cis}\,(\frac{3\pi}{5})$,

$z_3 = \text{cis}\,\pi = -1$ (real root),

$z_4 = \text{cis}\,(\frac{7\pi}{5})$, $z_5 = \text{cis}\,(\frac{9\pi}{5})$

Now $z_4 = \text{cis}\,(\frac{7\pi}{5}) = \text{cis}\,(-2\pi + \frac{7\pi}{5})$

$\therefore$ $z_4 = \text{cis}\,(-\frac{3\pi}{5}) = \bar{z}_2$

Similarly $z_5 = \bar{z}_1$

The quadratic factor given by $(z - z_1)(z - z_5)$

$= z^2 - (z_1 + z_5)z + z_1z_5$

$= z^2 - (z_1 + \bar{z}_1)z + z_1\bar{z}_1$

$= z^2 - (2\cos\frac{\pi}{5})\,z + 1$

Similarly the quadratic factor given by $(z - z_2)(z - z_4)$ is

$z^2 - (2\cos\frac{3\pi}{5})\,z + 1$

$\therefore$ $z^5 + 1 = (z + 1)[(z - z_1)(z - z_5)]$
$[(z - z_2)(z - z_4)]$

$$= (z + 1)\Big[z^2 - (2\cos\frac{\pi}{5})\,z + 1\Big]$$

$$\Big[z^2 - (2\cos\frac{3\pi}{5})\,z + 1\Big]$$

(c) $z^{15} + 1 = (z^3)^5 + 1$

Let $w = z^3$, then $z^{15} + 1 = w^5 + 1$

$= (w + 1)(w^4 + w^3 + w^2 + w + 1)$ (1)

$w^4 + w^3 + w^2 + w + 1 =$

Continued

$$\left[w^2 - \left(2\cos\frac{\pi}{5}\right)w + 1\right] \times$$

$$\times \left[w^2 - \left(2\cos\frac{3\pi}{5}\right)w + 1\right]$$

$$= \left(z^6 - 2z^3\cos\frac{\pi}{5} + 1\right)\left(z^6 - 2z^3\cos\frac{3\pi}{5} + 1\right)$$

Hence from (1) the required result follows.

Solution 4.7(2): $1 + z + z^2 + z^3 + z^4 + z^5$ is a geometric series with the first term equal to 1 and the common ratio z.

$$\therefore \quad 1 + z + z^2 + z^3 + z^4 + z^5 = \frac{z^6 - 1}{z - 1} \quad (1)$$

To solve $z^6 - 1 = 0$, we have:

$z^6 = 1 = \text{cis}\,(2k\pi)$

The six roots of this equation are:

$$z = \text{cis}\left(\frac{k\pi}{3}\right),\ k = 0, 1, 2, 3, 4, 5$$

The real root for k = 0 is z = 1. Now from (1)

$z^6 - 1 = (z - 1)(z^5 + z^4 + z^3 + z^2 + z + 1)$

So the six roots of $z^6 - 1$ are also given by

z = 1 and by those of

$z^5 + z^4 + z^3 + z^2 + z + 1 = 0.$

Hence the five roots of the last equation are

given by $z = \text{cis}\left(\frac{k\pi}{3}\right)$, k = 1, 2, 3, 4, 5

These are: $z_1 = \text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$,

$z_2 = \text{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z_3 = -1,$

$z_4 = \text{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$

$z_5 = \text{cis}\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution 4.7(3): $z^4 + 1 = z^4 + 2z^2 + 1 - (\sqrt{2}z)^2$

$= (z^2 + 1)^2 - (\sqrt{2}z)^2$

$= (z^2 + 1 + \sqrt{2}z)(z^2 + 1 - \sqrt{2}z)$

$= (z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1)$

We can now solve $z^4 + 1 = 0$.

Using the factors of $z^4 + 1$,

$(z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1) = 0$

$\therefore \quad z^2 + \sqrt{2}z + 1 = 0$ gives $z = \dfrac{-\sqrt{2} \pm \sqrt{2}i}{2}$

and $z^2 - \sqrt{2}z + 1 = 0$ gives $z = \dfrac{\sqrt{2} \pm \sqrt{2}i}{2}$

Hence the four roots of $z^4 + 1 = 0$ are:

$$\frac{-\sqrt{2} \pm \sqrt{2}i}{2},\ \frac{\sqrt{2} \pm \sqrt{2}i}{2}$$

By De Moivre's theorem:

$z^4 = -1 = \text{cis}\,\pi = \text{cis}\,(\pi + 2k\pi)$

Let $z = r\,\text{cis}\,\theta$ be a root of $z^4 + 1 = 0$. By De Moivre's theorem: $\quad z^4 = r^4\,\text{cis}\,4\theta$

Equating the moduli of two equal complex numbers given by z^4 we now have:

$r^4 = 1$, i.e. $r = 1$ and $4\theta = \pi + 2k\pi$,

$$\theta = \frac{2k\pi + \pi}{4}, \quad k = 0, 1, 2, 3$$

The four roots are:

$$z_1 = \text{cis}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$z_2 = \text{cis}\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$z_3 = \text{cis}\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$z_4 = \text{cis}\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Solution 4.7(4): $z^4 - 2z^2 + 4 = 0$

We solve this as a quadratic in (z^2)

$$\therefore \quad z^2 = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$$

$$\therefore \quad z = \pm\sqrt{1 \pm \sqrt{3}i} \quad (1)$$

To find $\sqrt{1 + \sqrt{3}i}$, let $w^2 = 1 + \sqrt{3}\,i$, where $w = x + iy$

$\therefore \quad x^2 - y^2 + 2ixy = 1 + \sqrt{3}\,i$

Equating real and imaginary parts:

$x^2 - y^2 = 1, \quad 2xy = \sqrt{3}$

$\therefore \quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 1 + 3 = 4$

$\therefore \quad x^2 + y^2 = 2$ since $x^2 + y^2 > 0$

By adding, $x^2 + y^2 = 2$, $x^2 - y^2 = 1$

$$2x^2 = 3 \text{ gives } x = \pm\sqrt{\frac{3}{2}}$$

$$\text{By subtracting, } 2y^2 = 1 \text{ gives } y = \pm\frac{1}{\sqrt{2}}$$

Considering $2xy = \sqrt{3}$ we combine $x = \sqrt{\frac{3}{2}}$

with $y = \frac{1}{\sqrt{2}}$ and $x = -\sqrt{\frac{3}{2}}$ with $y = -\frac{1}{\sqrt{2}}$

$$\text{Hence } \sqrt{1 + \sqrt{3}i} = \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i$$

The conjugate root, similarly is

$$\sqrt{1 - \sqrt{3}i} = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i$$

Hence the required roots are

$$\pm\left(\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i\right), \pm\left(\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i\right)$$

Note: We can also use De Moivre's theorem to find $\sqrt{1 + \sqrt{3}i}$.

Continued

Now $1 + \sqrt{3}i = 2\text{cis}\left(\frac{\pi}{3}\right) = 2\text{cis}\left(2k\pi + \frac{\pi}{3}\right)$

$$\therefore \quad \sqrt{1 + \sqrt{3}i} = \sqrt{2}\,\text{cis}\left(k\pi + \frac{\pi}{6}\right),\ k = 0, 1$$

$$\text{i.e. } \sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i$$

$$\text{and } \sqrt{2}\,\text{cis}\left(\frac{5\pi}{6}\right) = \sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= -\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i$$

The conjugate roots are found similarly.

Solution 4.7(5): $z^6 + 2z^3 + 2 = 0$

Let $w = z^3$, then $w^2 + 2w + 2 = 0$

$$\therefore \quad w = z^3 = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$$

We now have: $z^3 = -1 + i \quad (1)$

and $z^3 = -1 - i \quad (2)$

Now $-1 + i = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

So, $z^3 = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4} + 2k\pi\right)$. The three roots of $z^3 = -1 + i$ are:

$$z = 2^{1/6}\,\text{cis}\left(\frac{8k\pi + 3\pi}{12}\right), \quad k = 0, 1, 2$$

Again $-1 - i = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

So from (2): $z^3 = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)$.

The three roots of $z^3 = -1 - i$ are:

$$z = 2^{1/6}\,\text{cis}\left(\frac{8k\pi - 3\pi}{12}\right),\ k = 0, 1, 2$$

Hence the six roots of $z^6 + 2z^3 + 2 = 0$ are:

$$z = 2^{1/6}\,\text{cis}\left(\frac{8k\pi \pm 3\pi}{12}\right),\ k = 0, 1, 2$$

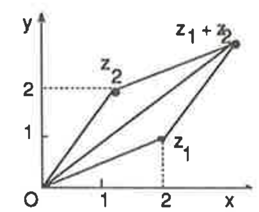
SOLUTIONS: EXERCISE 4.8

Solution 4.8(1):

(a) $z_1 + z_2$

$= 2 + i + 1 + 2i$

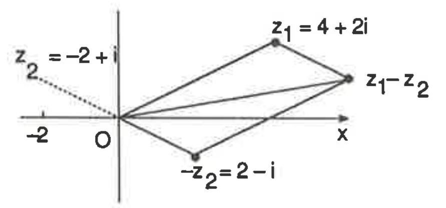
$= 3 + 3i$



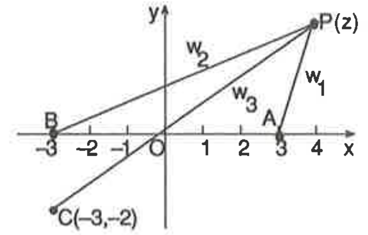
(b) $z_1 - (z_2)$

$= 4 + 2i - (-2 + i)$

$= 6 + i$



Solution 4.8(2):



Solution 4.8(3): OPQR is a square. P is represented by the complex number

$z = 2\,\text{cis}\,60° = 1 + \sqrt{3}\,i$

We find $R(z_2)$ before $Q(z_1)$.

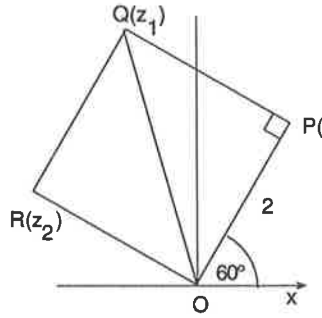
$z_2 = i\cdot z \quad (\angle ROP = 90°)$

$= i\cdot(1 + \sqrt{3}\,i)$

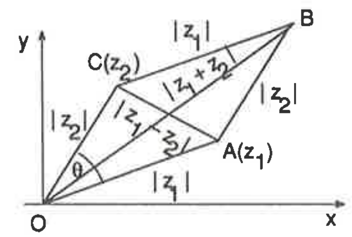
$\therefore \quad z_2 = -\sqrt{3} + i$

Then $z_1 = z + z_2 = 1 + \sqrt{3}\,i + (-\sqrt{3} + i)$

$\therefore \quad z_1 = (1 - \sqrt{3}) + (1 + \sqrt{3})\,i$



Solution 4.8(4):



Using $|w|^2 = w\bar{w}$ and $\overline{w_1 \pm w_2} = \bar{w}_1 \pm \bar{w}_2$

$|z_1 - z_2|^2 + |z_1 + z_2|^2$

$= (z_1 - z_2)\overline{(z_1 - z_2)} + (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1(\bar{z}_1 - \bar{z}_2 + \bar{z}_1 + \bar{z}_2)$

$\quad + z_2(-\bar{z}_1 + \bar{z}_2 + \bar{z}_1 + \bar{z}_2)$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2$

Alternative Method:

Let the points A and B in the complex plane represent the complex numbers z_1 and z_2 respectively. Complete the parallelogram OABC.

B represents $z_1 + z_2$

$\therefore$ Length OB $= |z_1+z_2|$ (1)

and length CA $= |z_1-z_2|$ (2)

Let $\angle AOC = \theta$, then $\angle OAB = 180° - \theta$

In triangle OAC:

$$AC^2 = OA^2 + OC^2 - 2OA\cdot OC\cdot \cos\theta$$

$$\therefore\ |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\cdot|z_2|\cdot\cos\theta \quad (3)$$

In Δ OAB:

$$OB^2 = AO^2 + AB^2 - 2AO\cdot AB\cdot \cos(180° - \theta)$$

$$\therefore\ |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\cdot|z_2|\cdot(-\cos\theta) = |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2|\cdot\cos\theta \quad (4)$$

Adding (3) and (4)

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

To give a geometrical meaning to this result, we write

$\therefore$ $|z_1+z_2| = OB$, $|z_1-z_2| = AC$

$|z_1| = OA$ and $|z_2| = OC$ then :

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

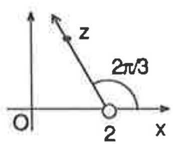
This result says that the sum of squares of the diagonals of a parallelogram is equal to the sum of squares of the four sides.

SOLUTIONS: EXERCISE 4.9

Solution 4.9(1):

(a) $\arg(z-2) = \frac{2\pi}{3}$

The locus of z is a ray starting at A(2, 0), excluding the point A. The equation of this ray is

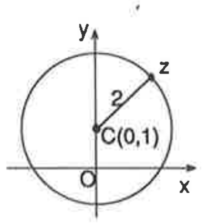


$$y = (x-2)\tan 120° = -\sqrt{3}(x-2), \quad y > 0$$

(b) $|z - i| = 2$ The locus of z is a circle of radius 2 with the centre at C(0, 1)

The Cartesian equation is:

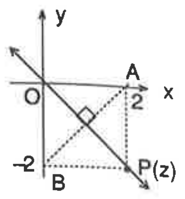
$$x^2 + (y-1)^2 = 4$$



(c) $|z-2| = |z+2i|$

Let P represent the complex number z in the complex plane.

A is (2,0), B is (0,−2)



$\therefore$ PA $= |z-2|$,

PB $= |z+2i|$

So $|z-2| = |z+2i|$ gives PA = PB for all positions of P.

$\therefore$ the locus of P is the perpendicular bisector of AB. Hence the locus of z is the perpendicular bisector of AB where A(2, 0) and B(0, −2). The cartesian equation is $y = -x$.

(d) Let $z = x + iy$, then $|z-2| = 2|z+i|$ becomes

$$|(x-2) + iy| = 2|x + i(y+1)|$$

$$\therefore\ |(x-2) + iy|^2 = 4|x + i(y+1)|^2$$

$$\therefore\ (x-2)^2 + y^2 = 4[x^2 + (y+1)^2]$$

This simplifies to:

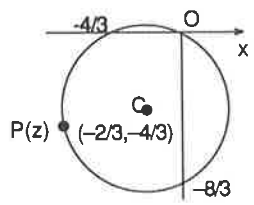
$$3x^2 + 3y^2 + 4x + 8y = 0 \quad (1)$$

Divide by 3, then

$$(x^2 + \tfrac{4}{3}x) + (y^2 + \tfrac{8}{3}y) = 0$$

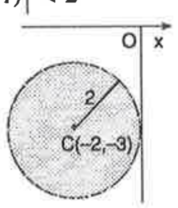
$$\therefore\ (x + \tfrac{2}{3})^2 + (y + \tfrac{4}{3})^2 = \frac{20}{9} \quad (2)$$

Hence the locus of z represents a circle of radius $\frac{2}{3}\sqrt{5}$ with the centre at $(-\frac{2}{3}, -\frac{4}{3})$



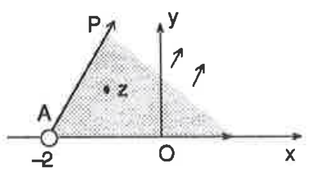
Solution 4.9(2):(a) $|z - (-2 - 3i)| < 2$

The locus of z is the set of points inside the circle of radius 2 with the centre at C(−2, −3). The points on the circle are excluded from the set.



(b) $0 \le \arg(z+2) \le \frac{\pi}{3}$

The locus of z is an angular region PAX where $\angle PAX = 60°$,



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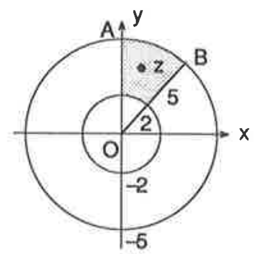
including the points on the arms of $\angle PAX$, but excluding the point A(−2, 0).

(c) $\left.\begin{array}{l} 2 \le |z| \le 5, \\ \pi/3 \le \arg z \le \pi/2 \end{array}\right\}$

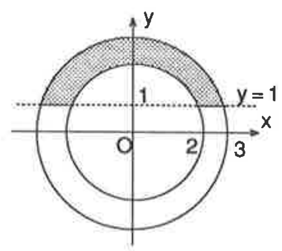
The locus of z is a region bounded by the arcs of two concentric circles with radii 2 and 5 and the centre at O(0, 0) and the arms of angles $\angle AOB$ where $\angle AOX = 90°$, $\angle BOX = 60°$.



(d) $2 \le |z| \le 3$ and $\text{Im}(z) \ge 1$

The locus of z is a region between two concentric circles of radii 2 and 3 with the centre at O(0, 0) and the line y = 1.

The points on the boundaries are included.



Solution 4.9(3): $w = z - \frac{1}{z}$, given $|z| = 2$

We have $w = z - \frac{1}{z} = z - \frac{\bar{z}}{z\bar{z}}$

Now $|z| = 2$, so $z\bar{z} = 4$

$\therefore$ $w = z - \frac{1}{4}\bar{z}$

Substituting $z = x + iy$, $w = u + iv$,

$$u + iv = x + iy - \frac{1}{4}(x - iy) = \frac{3}{4}x + \frac{5}{4}yi$$

Equating the real and the imaginary parts:

$$u = \frac{3x}{4} \text{ and } v = \frac{5y}{4}$$

We eliminate x and y as follows:

$$x = \frac{4u}{3} \text{ and } y = \frac{4v}{5}$$

Now $|z| = 2$ (given)

$\therefore$ $x^2 + y^2 = 4$, $\frac{16}{9}u^2 + \frac{16}{25}v^2 = 4$

$\therefore$ $100u^2 + 36v^2 = 225$

Hence the locus of w is an ellipse $100x^2 + 36y^2 = 225$

Solution 4.9(4): $w = \frac{z}{z+2}$, $\arg w = \frac{\pi}{6}$

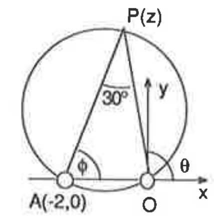
We have $w = \frac{z}{z+2}$

$\therefore$ $\arg w = \arg(\frac{z}{z+2}) = \arg z - \arg(z+2)$

$\therefore$ $\arg z - \arg(z+2) = \frac{\pi}{6}$

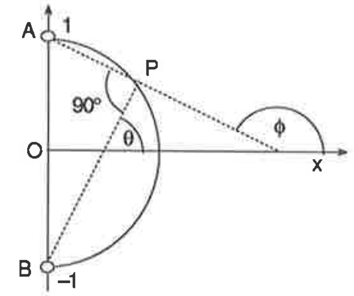
$\therefore$ $\angle POX - \angle PAX = \frac{\pi}{6}$ $\therefore$ $\angle APO = \frac{\pi}{6}$

Hence the locus of z as represented by P is thus the major arc of a circle with AO as a chord which subtends an angle of $\frac{\pi}{6}$ at the circumference. The points A and O are excluded from the locus.



Solution 4.9(5):

(a) $\arg(z-i) - \arg(z+i) = \frac{\pi}{2}$



Arg $(z + i)$ is the angle θ made by BP with the positive x-axis, where B is (0,−1) and P(z) any point on the locus.

Arg$(z - i)$ is the angle ϕ made by PA with the x-axis, where A is (0, 1).

We have $\arg(z-i) - \arg(z+i) = \frac{\pi}{2}$

$\therefore$ $\phi - \theta = \frac{\pi}{2}$

$\therefore$ $\angle APB = 90°$.

Thus the locus of P(z) is the semi-circle with AB as the diameter such that x > 0.

(b) Let $z = x + iy$

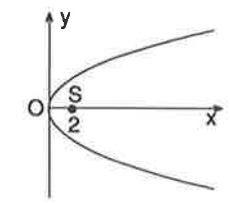
We have $\text{Re}(z+2) = |z-2|$

$\therefore$ $x + 2 = \sqrt{(x-2)^2 + y^2}$

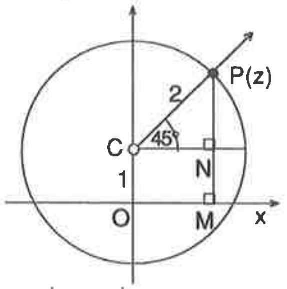
$(x+2)^2 = (x-2)^2 + y^2$

This simplifies to $y^2 = 4x$.

Hence the locus of z is the parabola with focus at S(2, 0) and the directrix x = −2.



Solution 4.9(6): $|z - i| = 2$ and $\arg(z - i) = \frac{\pi}{4}$



The locus of $|z - i| = 2$ is a circle of radius 2 with the centre at $C(0, 1)$.
Arg $(z - i)$ represents a ray through C, making an angle of 45°. These two loci intersect in only one point $P(x, y)$.

$CP = 2, \angle PCN = 45°$

$x = OM = CN = 2\cos 45° = \sqrt{2}$

$y = PM = PN + NM = 2 \sin 45° + 1$

$= \sqrt{2} + 1$

$\therefore$ P is $(\sqrt{2}, \sqrt{2} + 1)$.

SOLUTIONS OF PROBLEMS: CHAPTER 5

SOLUTIONS: EXERCISE 5.1

Solution 5.1(1): (a) $16x^4 - 1$

$= (4x^2 - 1)(4x^2 + 1)$

$= (2x - 1)(2x + 1)(2x + i)(2x - i)$

(b) $4x^2 + x + 3 = 4\left(x^2 + \frac{x}{4} + \frac{3}{4}\right)$

$= 4\left[\left(x + \frac{1}{8}\right)^2 + \frac{47}{64}\right]$

$= 4\left(x + \frac{1}{8} + i\frac{\sqrt{47}}{8}\right)\left(x + \frac{1}{8} - i\frac{\sqrt{47}}{8}\right)$

(c) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$

$= (x^2 + 1)^2 - x^2$

$= (x^2 + x + 1)(x^2 - x + 1)$

$= \left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]\left[\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right]$

$= \left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \times$

$\times \left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

Solution 5.1(2): We use the Remainder theorem: When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

$P(x) = 2x^4 + x^2 - i = x^2(2x^2 + 1) - i$

When $P(x)$ is divided by $x - (1 - 2i)$ the remainder is $P(1 - 2i)$.

Now $x = 1 - 2i$

$\therefore \quad x^2 = 1 - 4i + 4i^2 = -3 - 4i$

$\therefore \quad P(1 - 2i) = (-3 - 4i)(-6 - 8i + 1) - i$

$= (3 + 4i)(5 + 8i) - i$

$= 15 + 44i + 32i^2 - i = -17 + 43i$

Solution 5.1(3): Let $P(x) = x^4 + bx^3 + cx^2 - x + 2$

Since $P(x)$ is divisible by $x^2 - 1$, it is divisible by each of the factors of $x^2 - 1$, i.e. $(x - 1)$ and $(x + 1)$. We use the Factor theorem: If $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$. Substituting $x = 1$

$P(1) = 1 + b + c - 1 + 2 = 0$

$\therefore \quad b + c = -2 \qquad (1)$

Again substituting $x = -1$,

$P(-1) = 1 - b + c + 1 + 2 = 0$

$\therefore \quad -b + c = -4 \qquad (2)$

Adding (1) and (2): $2c = -6$, giving $c = -3$

Then from (1): $b - 3 = -2$, giving $b = 1$

Hence $b = 1$ and $c = -3$.

Solution 5.1(4): When $P(x)$ is divided by $(x - 3)(x - 4)$, the remainder must be of the form $ax + b$.

$\therefore \quad P(x) = (x - 3)(x - 4)Q(x) + ax + b \quad (1)$

By the Remainder theorem: When $P(x)$ is divided by $(x - c)$, the remainder is $P(c)$.

$\therefore \quad P(3) = 0 + 3a + b = 3$

and $P(4) = 0 + 4a + b = 4$

$\therefore \quad 3a + b = 3 \qquad (2)$

$4a + b = 4 \qquad (3)$

Subtracting (2) from (3): $a = 1$

Then from (2), $b = 0$.

So, from (1), when $P(x)$ is divided by $(x - 3)(x - 4)$, the remainder is x.

Solution 5.1(5):

Let $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$

$\therefore \quad P'(x) = 4x^3 - 18x^2 + 24x - 10$

$P''(x) = 12x^2 - 36x + 24 = 12(x - 1)(x - 2)$

Since $P(x) = 0$ has a root of multiplicity 3, $P'(x)$ has a zero of multiplicity 2 and $P''(x)$ has a common zero with $P(x)$.

Now $P''(x) = 12(x - 1)(x - 2)$

$P(1) = 1 - 6 + 12 - 10 + 3 = 0$

$P(2) = 16 - 48 + 48 - 40 + 3 \neq 0$

Hence $x = 1$ is the triple zero of $P(x)$.

$\therefore \quad P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3 \quad (1)$

$P(x) = (x - 1)^3(ax + b)$

$= (x^3 - 3x^2 + 3x - 1)(ax + b) \quad (2)$

Comparing the coefficients of x^3 and the constant terms of (1) and (2):

$a = 1, b = -3$

Hence the roots of $P(x) = 0$ are: $x = 1, 1, 1, 3$.