SELF-TESTING EXERCISE 4.1 (For complete solutions see page 233.)

1. Write the following as imaginary numbers: (a) $3\sqrt{-25}$ (b) $3\sqrt{-\frac{1}{4}}$ (c) $\sqrt{-12}$

2. Simplify: (a) $(3 + i) + (-2 - 3i)$ (b) $(2 - i) - (5 - 2i)$ (c) $(\frac{1}{2} - \frac{i}{3}) + (\frac{1}{3} + \frac{i}{2})$

3. Multiply and answer in the form a + ib

   (a) $\sqrt{-3} \times \sqrt{-12}$ (b) $(2i)(3i)$ (c) $(1 - i)(1 + i)$ (d) $(2 - 3i)^2$

4. Divide and answer in the form a + ib

   (a) $\dfrac{2 - 3i}{1 + 2i}$ (b) $\dfrac{-5 - 4i}{-1 + i}$ (c) $\dfrac{3 - 2i}{5i}$ (d) $\dfrac{1 + 2i}{i^3}$

5. Find x and y if $(x - iy)^2 = -2\sqrt{3} - 2i$

6. If $z = x + iy$, express $\dfrac{z + 1}{z - 1}$ in the form a + ib.

7. Find the quadratic equation whose roots are $2 + i$ and $\dfrac{1}{2 + i}$

8. Solve the following for z: $\dfrac{1}{z} = 1 + i + \dfrac{2}{1 - i}$

9. Solve the following pair of equations for z and w:

   $(2 + i)z + (2 - i)w = 1$ and $(2 - i)z + (2 + i)w = 2$

10. Given $z = 1 + i$, find: (a) z^2 (b) z^4 (c) $\dfrac{1}{z^2} + z^2$

SELF-TESTING EXERCISE 4.2 (For complete solutions see page 234.)

1. Express each of the following complex numbers in the mod–arg form:

   (a) $z = 4$, (b) $z = -4$, (c) $z = 4i$, (d) $z = 2 + 2i$, (e) $z = \dfrac{\sqrt{3}}{2} - \dfrac{i}{2}$

2. Express the following in the form a + ib:

   (a) $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ (b) $2[\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})]$

3. Simplify each of the following:

   (a) $[2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})]^2 \times [5(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})]^2$

   (b) $[2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})]^2 \div [5(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})]^2$

4. Express $\sqrt{3} + i$, $1 - i$, $1 + \sqrt{3}\,i$, $\sqrt{3} - i$ in mod–arg form and find

   $\dfrac{(\sqrt{3} + i)(1 - i)}{(1 + \sqrt{3}\,i)(\sqrt{3} - i)}$ in the r cis θ form.

5. Prove that $[r(\cos\theta + i\sin\theta)]^3 = r^3(\cos 3\theta + i\sin 3\theta)$

SELF-TESTING EXERCISE 4.3 (For complete solutions see page 235.)

1. If $z = 1 + i$, evaluate z^8 and z^{-8}

2. Simplify to the form $r \text{ cis } \theta$: $\dfrac{[3(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6})]^6}{[2(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3})]^2}$

3. Simplify to the form $r \text{ cis } \theta$: $(1 + \sqrt{3}\, i)^3 \div (1 - i)^4$

4. Express (a) $\cos 5\theta$, (b) $\sin 5\theta$, in terms of $\cos\theta$ and $\sin\theta$ and hence express $\tan 5\theta$ in terms of $\tan\theta$.

5. Express $\cos^6\theta$ in terms of cosines of multiples of θ, hence find $\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta$

6. Show that $(1 + \cos 2\theta + i \sin 2\theta)^n = 2^n \cos^n\theta(\cos n\theta + i \sin n\theta)$

7. Prove that $1 + z + z^2 + z^3 = \dfrac{1-z^4}{1-z}$. Use this result and $z = \cos\theta + i\sin\theta$

   to prove that: $1 + \cos\theta + \cos 2\theta + \cos 3\theta = \dfrac{1}{2}\left(1 + \dfrac{\sin 7\theta/2}{\sin \theta/2}\right)$

8. Find the modulus and argument of z where $z = \dfrac{\sqrt{3}+i}{1+i}$. Find the smallest positive integer n such that z^n is real, hence evaluate z^n for this value of n.

9. Prove that $\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ and hence deduce that

   $$\left(1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0$$

10. Find the smallest positive integer m such that:

    $$(\sqrt{3} + i)^m = (\sqrt{3} - i)^m$$

SELF-TESTING EXERCISE 4.4 (For complete solutions see page 237.)

1. Find the square roots of: (a) i, (b) $2 + 2i$
2. Find the three cube roots of $\sqrt{3} + i$ and indicate them on an Argand diagram. Find the area of the triangle formed by the points representing these roots.
3. Solve for z in the form $a + ib$: $z^2 - (2 - i)z - 2i = 0$
4. Find all the six roots of $z^6 = -i$ and show these on an Argand diagram.

SELF-TESTING EXERCISE 4.5 (For complete solutions see page 238.)

1. Show that: (a) $\overline{\bar{z} + ai} = z - ai$, (a is real),

(b) $\overline{\dfrac{(2+i)^2}{3-4i}} = 1$, (c) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2 \bar{z}_3}$

2. If $z = x + iy$ and $w = 1 + i$, prove that $(z - w)(\bar{z} - \bar{w}) = 1$ represents a circle of radius 1 with the centre at the point (1, 1). (Hint: Expand and substitute.)

3. (a) If $z = \cos\theta + i\sin\theta$, prove that $\dfrac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$,

(b) If $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$, $z_3 = \cos\gamma + i\sin\gamma$ and

$z_1 + z_2 + z_3 = 0$, then prove that $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = 0$.

(Hint: Use part (a) to transform $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}$ to $\bar{z}_1 + \bar{z}_2 + \bar{z}_3$.)

4. (a) Prove that $1 - \cos\theta + 2i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2} + 2i\cos\frac{\theta}{2}\right)$

(b) Prove that $(1 - \cos\theta + 2i\sin\theta)^{-1} = \dfrac{1 - 2i\cot\left(\frac{\theta}{2}\right)}{5 + 3\cos\theta}$

(Hint: Use (a), then rationalise the denominator.)

5. Solve the equation for z: $z^2 = i\bar{z}$

SELF-TESTING EXERCISE 4.6 (For complete solutions see page 240.)

1. ω is the complex root of $z^6 - 1 = 0$ with the smallest positive argument.
   (a) Prove that ω, ω^2, ω^4 and ω^5 are the roots of $z^4 + z^2 + 1 = 0$.
   (b) Find the quadratic equation whose roots are $\alpha = \omega + \omega^5$ and $\beta = \omega^2 + \omega^4$.

2. If $z = a + ib$ is a solution of $z^3 - 1 = 0$, prove that $z = a - ib$ is also a solution.

3. (a) Solve $z^5 - 1 = 0$ by De Moivre's theorem. Show these roots on an Argand diagram..
   (b) Prove that $(z - 1)(z^4 + z^3 + z^2 + z + 1) = z^5 - 1$.
   (c) Solve the equation: $z^4 + z^3 + z^2 + z = -1$.
   (d) Deduce that $\cos\dfrac{2\pi}{5}$ and $\cos\dfrac{4\pi}{5}$ are the roots of $4c^2 + 2c - 1 = 0$.
   (e) Prove that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$.

SELF-TESTING EXERCISE 4.7 (For complete solutions see page 241.)

1. (a) Factorise $z^3 + 1$ into linear and quadratic factors.

   (b) Prove that $z^5 + 1 = (z+1)\left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]\left[z^2 - 2\cos\left(\frac{3\pi}{5}\right)z + 1\right]$

Continued

   (c) Prove that $z^{15} + 1 = (z^3+1)(z^{12} - z^9 + z^6 - z^3 + 1)$

$$= (z+1)(z^2 - z + 1)\left(z^6 - 2z^3\cos\frac{\pi}{5} + 1\right)\left(z^6 - 2z^3\cos\frac{3\pi}{5} + 1\right)$$

2. Prove that $\dfrac{z^6 - 1}{z - 1} = z^5 + z^4 + z^3 + z^2 + z + 1$ and hence solve the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$

3. Show that $z^4 + 1 = (z^2 + \sqrt{2}\,z + 1)(z^2 - \sqrt{2}\,z + 1)$, hence solve $z^4 + 1 = 0$ by two methods.

4. Solve $z^4 - 2z^2 + 4 = 0$ $\left(\text{Hint: use } z^2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$

5. Find the six roots of $z^6 + 2z^3 + 2 = 0$.

SELF-TESTING EXERCISE 4.8 (For complete solutions see page 243.)

1. Find graphically: (a) $z_1 + z_2$ given $z_1 = 2 + i$ and $z_2 = 1 + 2i$
   (b) $z_1 - z_2$ given $z_1 = 4 + 2i$ and $z_2 = -2 + i$
2. Given $P(z = x + iy)$ in an Argand diagram, draw the vectors
   (a) $w_1 = z - 3$, (b) $w_2 = z + 3$, (c) $w_3 = z + 3 + 2i$, ending at P.
3. OPQR is a square in an Argand diagram where O is the origin. P represents
   $z = 2(\cos 60° + i \sin 60°)$.
   Find the complex numbers represented by Q and R.
4. Prove geometrically: $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$. What is the geometrical meaning of this result?

SELF-TESTING EXERCISE 4.9 (For complete solutions see page 244.)

1. Describe the locus of z represented by the following equations. Sketch and find the Cartesian equations of these loci:

   (a) $\arg(z-2) = \frac{2\pi}{3}$, (b) $|z - i| = 2$, (c) $|z - 2| = |z + 2i|$,

   (d) $|z - 2| = 2|z + i|$.

2. Sketch the regions in the complex plane which satisfy the following conditions:

   (a) $|z + 2 + 3i| < 2$, (b) $0 \le \arg(z+2) \le \frac{\pi}{3}$,

   (c) $2 \le |z| \le 5$ and $\frac{\pi}{3} \le \arg z \le \frac{\pi}{2}$, (d) $2 \le |z| \le 3$ and $\operatorname{Im}(z) \ge 1$.

3. Find the locus of w, where z is restricted as indicated:

   $w = z - \frac{1}{z}$ if $|z| = 2$.

4. Find the locus of z if $w = \frac{z}{z+2}$ and $\arg w = \frac{\pi}{6}$.

5. Sketch on an Argand diagram the locus of z described by the equations:

   (a) $\arg(z-i) - \arg(z+i) = \frac{\pi}{2}$, (b) $\operatorname{Re}(z+2) = |z - 2|$.

6. Sketch the loci given by each of the two equations $|z - i| = 2$ and $\arg(z-i) = \frac{\pi}{4}$. Find the complex number that satisfies both of them.