4.2

8. Consider the polynomial $P_1(z) - P_2(z)$.

If $P_1(z) = P_2(z)$ for more than n values of z, then $P_1(z) - P_2(z) = 0$ and $P_1(z) - P_2(z)$ has more than n zeros.

However, $\deg(P_1 - P_2) \leq n$.

$P_1(0) - P_2(0) = 0$

$\therefore b_0 - c_0 = 0$

$\Rightarrow b_0 = c_0$

$P_1'(0) - P_2'(0) = 0$ {see Pg 119 of text}

$\therefore c_1 - b_1 = 0$

$\therefore c_1 = b_1$

Continuing this pattern, we have

$P_1^n(0) - P_2^n(0) = 0$

$b_n - c_n = 0$

$\therefore b_n = c_n$

Similarly $P_1^{n-1}(0) - P_2^{n-1}(0) = 0 \Rightarrow b_{n-1} - c_{n-1} = 0$

$\therefore b_{n-1} = c_{n-1}$

4.2

12. $P(x) = x^4 + bx^3 + cx^2 + dx + 3$

Integer coefficients $\Rightarrow$ real coefficients.

$\therefore$ non-real roots occur in complex conjugate pairs.

$\therefore P(i) = 0 \;;\; P(-i) = 0$

$\Rightarrow (x-i)$ and $(x+i)$ are factors.

$\therefore (x-i)(x+i)$ is a factor

$\Rightarrow (x^2+1)$ is a factor

Hence, $P(x) = (x^2+1)(x-\alpha)(x-\beta)$

$\alpha = \dfrac{p}{q} \Rightarrow$ p is a divisor of constant term
q is a divisor of leading coeff.

$\therefore p = \pm 3 \;;\; q = \pm 1$ are possible values

$\therefore \alpha = \pm 3$ are the two possible values.

Constant term is 3 $\Rightarrow 1\alpha\beta = 3$

$\therefore \alpha = 3, \beta = 1$ or $\alpha = -3, \beta = -1$

But sum of roots is < 0

$\therefore \alpha = -3, \beta = -1$

$\Rightarrow P(x) = (x^2+1)(x+3)(x+1)$ over $\mathbb{R}$

Q13, 14 are similar.