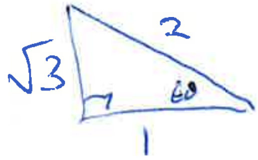
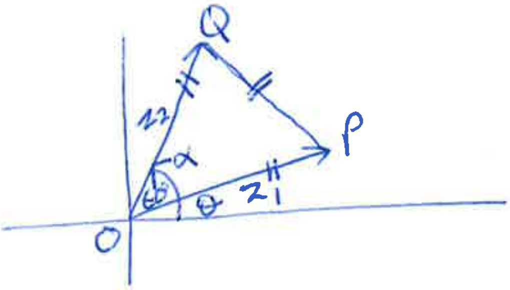
2.3



5. Method 1: Long way



ΔOPQ equilateral

$\therefore |z_1| = |z_2|$ and $\angle POQ = \frac{\pi}{3}$

Let $z_1 = r\,\mathrm{Cis}\,\theta$; $z_2 = r\,\mathrm{Cis}\,\alpha$

and $\alpha - \theta = \frac{\pi}{3}$

Show that $z_1^2 + z_2^2 = z_1 z_2$

$z_1^2 = r^2\,\mathrm{Cis}\,2\theta$; $z_2^2 = r^2\,\mathrm{Cis}\,2\alpha \rightarrow z_2^2 = r^2\,\mathrm{Cis}\,2(\theta + \frac{\pi}{3})$

$$\boxed{z_1 z_2 = r^2\,\mathrm{Cis}\,(\theta + \alpha)}$$

Key idea in this topic. When z_1 is multiplied by z_2, z_1 is rotated by $\arg(z_2)$ and scaled by a factor of $|z_2|$.

$$= r^2\,\mathrm{Cis}\left(\theta + \theta + \frac{\pi}{3}\right)$$
$$= r^2\,\mathrm{Cis}\left(2\theta + \frac{\pi}{3}\right)$$

$$\text{LHS} = z_1^2 + z_2^2$$
$$= r^2\,\mathrm{Cis}\,2\theta + r^2\,\mathrm{Cis}\left(2\theta + \frac{2\pi}{3}\right)$$
$$= r^2(\cos 2\theta + i\sin 2\theta) + r^2\left[\cos\left(2\theta + \frac{2\pi}{3}\right) + i\sin\left(2\theta + \frac{2\pi}{3}\right)\right]$$
$$= r^2(\cos 2\theta + i\sin 2\theta) + r^2\left[\cos 2\theta \cos\frac{2\pi}{3} - \sin 2\theta \sin\frac{2\pi}{3} + i\left(\sin 2\theta\cos\frac{2\pi}{3} + \cos 2\theta \sin\frac{2\pi}{3}\right)\right]$$
$$= r^2(\cos 2\theta + i\sin 2\theta) + r^2\left[\cos 2\theta \times -\frac{1}{2} - \sin 2\theta \times \frac{\sqrt{3}}{2} + i\left(\sin 2\theta \times -\frac{1}{2} + \cos 2\theta \times \frac{\sqrt{3}}{2}\right)\right]$$
$$= r^2\left(\cos 2\theta - \frac{1}{2}\cos 2\theta - \sin 2\theta \times \frac{\sqrt{3}}{2}\right) + i\left[\sin 2\theta - \frac{1}{2}\sin 2\theta + \cos 2\theta \times \frac{\sqrt{3}}{2}\right]$$
$$= r^2\left(\frac{1}{2}\cos 2\theta - \sin 2\theta \times \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2}\sin 2\theta + \cos 2\theta \times \frac{\sqrt{3}}{2}\right)$$
$$= r^2\left(\cos\frac{\pi}{3}\cos 2\theta - \sin 2\theta \times \sin\frac{\pi}{3}\right) + i\left(\cos\frac{\pi}{3}\sin 2\theta + \cos 2\theta \times \sin\frac{\pi}{3}\right)$$

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5. ctd

$= r^2\left[\cos\left(2\theta+\frac{\pi}{3}\right)\right] + i\left[\sin\left(2\theta+\frac{\pi}{3}\right)\right]$

$= r^2 \operatorname{cis}\left(2\theta+\frac{\pi}{3}\right)$

$= z_1 z_2$

$=$ RHS

Method 2: short way.

$z_2 =$ anticlockwise rotation of z_1 by $\frac{\pi}{3}$ about O from diagram and scaled by a factor 1.

$\therefore z_2 = \operatorname{cis}\frac{\pi}{3} z_1$

LHS

$\therefore = z_1^2 + z_2^2$

$= z_1^2 + \left(\operatorname{cis}\frac{\pi}{3} z_1\right)^2$

$= z_1^2 + \operatorname{cis}\frac{2\pi}{3} z_1^2$

$= z_1^2\left(1 + \operatorname{cis}\frac{2\pi}{3}\right)$

$= z_1^2\left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= z_1^2\left(1 - \frac{1}{2} + i\sin\frac{\pi}{3}\right)$

$= z_1^2\left(\frac{1}{2} + i\sin\frac{\pi}{3}\right)$

$= z_1^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= z_1^2 \operatorname{cis}\frac{\pi}{3} = z_1 \times z_1 \operatorname{cis}\frac{\pi}{3}$

$= z_1 z_2$

QED.

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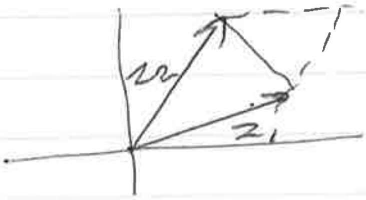
Q6. Recall the triangle inequality



$$|z_1 + z_2| \leq |z_1| + |z_2|$$

with equality $\Longleftrightarrow z_1 = k z_2$

$k \in \mathbb{R}$ and $k > 0$

<u>case 1:</u>

If $z_1 = z_2 = 0$, $\big||z_1| - |z_2|\big| = |z_1 + z_2|$

<u>case 2:</u> Let $z_1 \neq 0$, $z_2 \neq 0$

We need to prove $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

Consider $|z_1| - |z_2|$. We can write this as $|z_1 + z_2 - z_2| - |z_2|$. This can be written as $|(z_1 + z_2) + (-z_2)| - |z_2|$

Now using the triangle inequality

$$\boxed{|(z_1 + z_2) + (-z_2)|} - |z_2| \leq \boxed{|z_1 + z_2| + |-z_2|} - |z_2|$$

$|-z_2| - |z_2| = 0$

This part comes from the triangle inequality.

We can now simplify this inequality to

$$|z_1| - |z_2| \leq |z_1 + z_2|$$

with equality $\Longleftrightarrow z_1 + z_2 = k(-z_2)$

$\therefore z_1 = -kz_2 - z_2 = -(k+1)z_2$

and $k > 0$

Similarly consider $|z_2| - |z_1|$

$$|z_2| - |z_1|$$

$$= |z_2 + z_1 - z_1| - |z_1|$$

$$= |(z_2 + z_1) + (-z_1)| - |z_1|$$

Using the triangle inequality

$$|(z_2+z_1) + (-z_1)| - |z_1| \leq |z_2 + z_1| + |-z_1| - |z_1|$$

Simplifying this, we get

$\therefore |z_2| - |z_1| \leq |z_2 + z_1|$ with equality

$\Longleftrightarrow$ $z_2 + z_1 = k(-z_1)$

"if and only if"

$$\therefore z_2 = -z_1 - kz_1$$

$$z_2 = z_1(-1-k)$$

$$\therefore z_1 = \frac{1}{-1-k} z_2$$

$$= \frac{-1}{k+1} z_2$$

$\therefore$ z_1 is of the form $-Kz_2$ where $k > 0$ and $k \in \mathbb{R}$

Hence $||z_1| - |z_2|| \leq |z_2 + z_1|$

with equality $\Longleftrightarrow$ $z_1 = -kz_2$, $k > 0$

or $z_1 = 0$ or $z_2 = 0$