2.2

4(a) $z_1 = r_1(\cos\theta + i\sin\theta)$; $\overline{z_1} = r_1(\cos\theta - i\sin\theta)$

$$= r_1\left[\cos(-\theta) + i\sin(-\theta)\right]$$

S	A
T	C

$\cos\theta = \cos(-\theta)$

$\sin(-\theta) = -\sin\theta$

$z_2 = r_2(\cos\alpha + i\sin\alpha)$; $\overline{z_2} = r_2(\cos\alpha - i\sin\alpha)$

$$= r_2\left[\cos(-\alpha) + i\sin(-\alpha)\right]$$

$z_1 z_2 = r_1 r_2\left[\cos(\theta+\alpha) + i\sin(\theta+\alpha)\right]$ using the properties of modulus and argument

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

similarly

$\therefore\ \overline{z_1 z_2} = r_1 r_2\left[\cos(-\theta-\alpha) + i\sin(-\theta-\alpha)\right]$

$\overline{z_1} \times \overline{z_2} = r_1\left[\cos(-\theta) + i\sin(-\theta)\right] r_2\left[\cos(-\alpha) + i\sin(-\alpha)\right]$

$$= r_1 r_2\left[\cos(-\theta)\cos(-\alpha) + i\left(\cos(-\theta)\sin(-\alpha) + \sin(-\theta)\cos(-\alpha)\right) - \sin(-\theta)\sin(-\alpha)\right]$$

$$= r_1 r_2\left[\cos(-\theta-\alpha) + i\sin(-\theta-\alpha)\right]$$

$$= \overline{z_1 z_2}$$

2.2

4(b) $z = r(\cos\theta + i\sin\theta)$; $\bar{z} = r(\cos\theta - i\sin\theta)$

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

Multiply numerator and denominator by $\cos\theta - i\sin\theta$

$$\therefore \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{1}{r} \times (\cos\theta - i\sin\theta)$$

$$\therefore \overline{\left(\frac{1}{z}\right)} = \frac{1}{r}(\cos\theta + i\sin\theta)$$

Now consider $\frac{1}{\bar{z}}$

$$\frac{1}{\bar{z}} = \frac{1}{r(\cos\theta - i\sin\theta)}$$

Multiply numerator and denominator by $\cos\theta + i\sin\theta$

$$\therefore \frac{1}{\bar{z}} = \frac{1}{r(\cos\theta - i\sin\theta)} \times \frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{1}{r}(\cos\theta + i\sin\theta)$$

$= \overline{\left(\frac{1}{z}\right)}$ as required.

2.2 Let $z_1 = r_1(\cos\theta + i\sin\theta)$; $z_2 = r_2(\cos\alpha + i\sin\alpha)$

4 (c) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2}$ This is a known property

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ This is a known property

$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta-\alpha) + i\sin(\theta-\alpha)\right]$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2}\left[\cos(\theta-\alpha) - i\sin(\theta-\alpha)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(-(\theta-\alpha)) + i\sin(-(\theta-\alpha))\right]$$

Now lets look at $\overline{z}_1$ and $\overline{z}_2$.

$\overline{z}_1 = r_1(\cos\theta - i\sin\theta)$; $\overline{z}_2 = r_2(\cos\alpha - i\sin\alpha)$

$\overline{z}_1 = r_1\left[\cos(-\theta) + i\sin(-\theta)\right]$; $\overline{z}_2 = r_2\left[\cos(-\alpha) + i\sin(-\alpha)\right]$

Note: $\begin{array}{c|c} S & A \\ \hline T & C \end{array} \Rightarrow \begin{array}{l} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{array}$

$\left|\frac{\overline{z}_1}{\overline{z}_2}\right| = \frac{|\overline{z}_1|}{|z_2|}$ This is a known property

$$= \frac{r_1}{r_2}$$

$$\arg\left(\frac{\overline{z}_1}{\overline{z}_2}\right) = \arg(\overline{z}_1) - \arg(\overline{z}_2) = -\theta - (-\alpha)$$
$$= \alpha - \theta$$

$$\therefore \frac{\overline{z}_1}{\overline{z}_2} = \frac{r_1}{r_2}\left[\cos(\alpha-\theta) + i\sin(\alpha-\theta)\right]$$

4(c) But $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{r_1}{r_2}\left[\cos(-(\theta-\alpha)) + i\sin(-(\theta-\alpha))\right]$

$= \frac{r_1}{r_2}\left[\cos(\alpha-\alpha) + i\sin(\alpha-\theta)\right]$

$= \frac{\overline{z_1}}{\overline{z_2}}$ as required.

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5. You won't be able to do this until you have studied Extension 1 Mathematical Induction.